Nonlinearities in Exchange-Rate Dynamics:

Chaos?

Vitaliy Vandrovych*

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Abstract***

Deterministic chaotic systems represent an appealing new way to model an economy, especially financial markets. They allow the generation of interesting dynamics without exogenous shocks and are unpredictable in the long run. In this paper, I test the hypothesis of chaotic dynamics in exchange rates by applying tools from dynamical systems theory. I find that exchange rate returns are highly nonlinear even when a GARCH-type process is fitted to the data and this result is compatible with chaos. But the calculation of two other prerequisites of chaotic dynamics, namely the correlation dimension and the maximum Lyapunov exponent, rejects the hypothesis of chaos in the data. I emphasize the importance of the proper implementation of chaos tests, as in limited data sets there is a tendency for downward bias in estimates of correlation dimension and this might lead to an incorrect conclusion that chaos is present in the data. As pure chaotic dynamics is not observed in the data, I stress the significance of considering a dynamic noise element in theoretical chaotic asset pricing models.

Keywords: Exchange rate dynamics, chaos, correlation dimension, Lyapunov exponents, BDS test

JEL classification: C22, C49, C51, C63, F31, G12

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* Ph.D. candidate, International Business School, Brandeis University, vivandr@brandeis.edu

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1. Introduction

Traditional structural models of exchange rate determination tend to perform poorly when confronted with data, both in in-sample and out-of-sample forecasting[1]. During the last two decades a number of nonstandard models of asset price determination were developed that attempt to accommodate the diversity of the economic agents and their limited access to information and capabilities of processing it when making economic decisions[2]. Simple models with heterogeneous agents frequently are successful in replicating many stylized facts observed in financial data, such as a large trading volume, volatility persistence, sudden swings in the market behavior or fat tails of return distributions (LeBaron, 2006).

In the majority of cases these models are highly nonlinear and result in a wide range of dynamic behavior, *including chaotic dynamics*, such as in De Grauwe et al (1993), Lux (1998), Brock and Hommes (1998), Da Silva (2001), Federici and Gandolfo (2002), De Grouwe and Grimaldi (2006) among others. The intellectual appeal of such models is that in many cases they do not require exogenous shocks to exhibit interesting, real-life like dynamics as it is generated endogenously through the interaction of agents, and evolutionary change of their strategies.

The possibility of chaotic dynamics in asset prices has a number of implications. As chaos is by definition a deterministic phenomenon, it implies that observed financial time series which exhibit highly nonlinear and erratic behavior are in fact a product of the deterministic system. One of the features of chaotic signals is that even if the dynamical system generating data is known, and starting points are available, only very short term forecasting is possible due to the sensitive dependence on the initial conditions (SDIC thereafter), - the hallmark of chaos. Very small mistakes in the measurement of initial conditions (and they are unavoidable in economics data) would result in completely useless predictions of the future states of the system. This certainly would invalidate any long term forecasting as such. At the same time, as a chaotic trajectory always stays on the

[1] Refer to Sarno and Taylor (2002: chapter 4) for detailed analysis of exchange rate determination theories and their empirical performance.
[2] See Cars Hommes (2006) for the excellent survey of this fast growing literature.

attractor[3] in the long run, it is possible to make predictions about the limits in which the economic system will evolve over time.

In this paper we single out one feature of the above mentioned literature, namely the possibility of chaotic dynamics, and test it empirically using the exchange rate data. Tools from dynamical systems theory, such as correlation dimension and maximum Lyapunov exponents are used to make distinction between data produced by random and deterministic systems[4]. Our analysis indicates that although exchange rate returns are not independently and identically distributed (iid thereafter) and contain some important nonlinear dependence, they do not possess the features that are required to classify them as chaotic.

As the testing procedures involve a number of subjective decisions on the selection of parameters and interpretation of the graphical information, we elaborate also on the importance of the proper implementation of chaos testing techniques. In particular, we show that the omission of Theiler correction[5] to the Grassberger and Procaccia algorithm may lead to an incorrect conclusion of chaotic dynamics in some variables.

Our results do not invalidate the theoretical literature that implies a chaotic origin of financial time series. Rather it indicates that some amount of stochastic ingredients (dynamic noise) has to be present in a model that attempts to replicate the observed financial data. While internal interactions of heterogeneous agents are important, and may serve as the skeleton of the model, exogenous "news" has to be taken into account as well, not necessarily as a driving force but as "noise", albeit small, that has important implications for the evolution of the economy.

The rest of the paper proceeds as following. In section 2 we provide short background information on the notion of chaos and methods used to test it in empirical data. Section 3 outlines the results of the previous studies on chaos in exchange rate series. Section 4 describes the data used in the paper. Section 5 gives details on testing methodology and results that we obtained. And section 6 concludes.

[3] Bounded although topologically strange set.
[4] See Brock (1986) for a rigorous mathematical introduction to the tests that help to distinguish between random and deterministic systems.
[5] This correction is considered as a norm in the physics empirical studies but is not widely used in economics papers.

2. Detecting chaos in empirical data: methodology

There is no unique definition of chaos, and the most common way to introduce it is by listing the important features that have to be present in any chaotic system. The signal is chaotic if it has all of the following attributes: it is nonlinear, has SDIC, a strange attractor, continuous broadband Fourier power spectrum, at least one positive Liapunov exponent and ergodicity[6]. Although any chaotic signal is produced by some purely deterministic system[7], visually it is indistinguishable from the stochastic one.

In this paper we study three of the above characteristics, namely: nonlinearity, by applying BDS test, the strange attractor, by calculating correlation dimension, and SDIC, by estimating the maximum Lyapunov exponent. The remaining part of this section elaborates on how to calculate these quantities using the empirical data.

2.1 General background and phase space reconstruction

Let's assume that the dynamics of an economy is generated by some unknown deterministic dynamical system. The set of all possible states of a system is called a *phase space*. Let's consider a finite-dimensional state space R^n, in which state of the system is characterized by a vector $\boldsymbol{s} \in R^n$. The dynamics of such a system may be represented either by *n*-dimensional map:

$$\boldsymbol{s}_{t+1} = F(\boldsymbol{s}_t) \tag{1}$$

or by a system of n first-order ordinary differential equations:

$$\frac{d}{dt}\boldsymbol{s}(t) = \boldsymbol{f}(\boldsymbol{s}(t)) \tag{2}$$

Given some initial conditions, the solution of the system (also called the *orbit* or *trajectory*) will be attracted to some subspace of the phase space, which is called *attractor* of the system, after a sufficiently long transition period. Typically, for dissipative systems[8]

[6] See De Grauwe et all (1993, pp.34-35), Abarbanel (1995) or Kantz and Schreiber (2004) for a detailed introduction to the subject.
[7] Mathematically given by some system of nonlinear difference or differential equations.
[8] Dissipative systems are systems for which volume in the phase space is usually contracted under the time evolution.

the volume that is filled by an attractor is much smaller than the volume of the phase space.

For some systems, motion within an attractor is unstable. It is expressed in an exponential separation of orbits of points that are close to each other on the attractor (Eckmann and Ruelle (1985))[9]. Attractors with such an instability property are called *strange attractors*, and correspondingly systems that possess them are called chaotic.

In empirical studies, the states of a dynamical system, $\boldsymbol{s}_t$, are not observed, and the deterministic equation of motion, $\boldsymbol{s}_{t+1} = F(\boldsymbol{s}_t)$, is not known. What is available to the researcher, is just some scalar time series of measurements, say N daily exchange rates $\{x_t\}_{t=1}^{N}$, which are related to $\boldsymbol{s}_t$ through the observer function, $x_t = h(\boldsymbol{s}_t)$, also not known. The situation can be even more complicated when some random measurement error[10] is also present:

$$x_t = h(\boldsymbol{s}_t) + g * e_t,\ e_t \approx IID(0,1), \tag{3}$$

where $h : R^n \to R$ is an observer function that maps the state of the original system to the scalar time series, g is noise level, and e_t stands for measurement error[11].

Knowledge of the precise initial conditions for the variables of the original system and knowledge of the functions *F* and *h* that represent the dynamics of the system would enable us to predict the exact state of the system at each point in time.

The embedding theorem of Takens (1981) provides the conditions under which the properties of original chaotic system may be reconstructed from the scalar time series of observations. The process of rebuilding the state vectors from the observed data is called a *phase space reconstruction*. This method involves creation of *m*-dimensional 'histories', $X_t = (x_t, x_{t+t}, ..., x_{t+(m-1)t})$, where *m* is the embedding dimension[12], and t is the time delay[13]. Takens proved that the resulting reconstructed trajectory, $X = (X_1, X_2, ..., X_M)'$,

[9] As an attractor normally is bounded, this special separation is evident only for short time periods until the boundary of the attractor is reached.
[10] This is also called observational noise.
[11] Here we follow the notations used in Bask M. (2002).
[12] A theoretically sufficient condition for *m* to be the embedding dimension is *m>2d*, where *d* is the actual topological dimension of the attractor, but in practice it is frequently the case that much lower *m* will work.
[13] One of the frequently used methods for the time delay is to set it equal to the first zero-crossing of the autocorrelation function. Another method uses the delay that corresponds to the first local minimum of the mutual information function (see Diks C. 1999, p.23).

with $M = N - (m-1)\ t$, has topological properties that are identical to the properties of the original system. Hence, it is possible to learn the dynamic characteristics of the unknown system by studying the dynamics of the reconstructed orbit.

Therefore, any study of chaotic dynamics in empirical data starts with the phase space reconstruction, followed then by application of tests that calculate the invariants of motion, such as fractal dimensions and maximum Lyapunov exponent. Finite (and low) fractal dimension is a necessary but not a sufficient condition for chaos. When it is combined with a positive maximum Lyapunov exponent, it represents a strong indication of chaotic origin of the data.

2.2 BDS test

As chaos is a nonlinear phenomenon, it is suggestive to find if any nonlinear dependence is present in the data before proceeding to more direct chaos tests. This task may be accomplished by the BDS test of Brock et al[14] (1987), which evaluates the null hypothesis that data is iid. Its rejection implies some kind of dependence that may come from a number of different models, stochastic and deterministic, linear and nonlinear. BDS statistics has an asymptotically standard normal distribution, and any dependence in the data will result in the test statistic being larger than the critical value for the standard normal distribution. In order to eliminate possible linear dependence, the time series is frequently pre-filtered by an autoregressive model, and the BDS test is applied to the residuals. If the null hypothesis is rejected even in this case, the presence of nonlinear dependence is established in the data.

Two major invariant quantities are used for the classification of chaotic systems: correlation dimension and the maximum Lyapunov exponent[15]. These invariants do not depend on the initial conditions or coordinate system in which the dynamics is studied, and are preserved under smooth nonlinear changes of coordinate system.

[14] See also Brock et al (1996) for the published version of the paper.
[15] See Abarbanel (1996), chapter 5, for detailed description.

2.3 Correlation dimension

As mentioned above, chaotic dynamical systems possess strange attractors. One of the features of strange attractors is that they have a fractal dimension which is smaller than the number of degrees of freedom of the system (Grassberger and Procaccia 1983a) (GP thereafter). *Hausdorff dimension* and *information entropy* used to be the most commonly applied measures of the attractor's structure. Papers by GP (1983a, 1983b) introduce a new measure which was named *correlation dimension*, which has since been predominantly used in empirical studies. In contrast to the old measures, correlation dimension is sensitive to the process of coverage of the attractor (how frequently different parts of the attractor are visited by the trajectory), and moreover it is computationally simpler and allows characterization of the attractor even for high-dimensional dynamical systems (GP 1983a).

Correlation dimension is based on the notion of correlation integral that measures spatial correlation of points on the attractor. An important contribution of GP is that they used result of Takens (1981), and showed how to calculate the correlation dimension given only the scalar time series. This procedure is described below.

Given the scalar time series, $\{x_t\}_{t=1}^{N}$, *m*-dimensional histories, $X_t = (x_t, x_{t+t}, \ldots, x_{t+(m-1)t})$, are created. Then this reconstructed orbit, $X = (X_1, X_2, \ldots, X_M)'$, with $M = N - (m-1)t$, may be used for calculation of the correlation integral, C(M,m,r), that is given by the following formula:[16]

$$C(M,m,r) = \frac{2}{M(M-1)} \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} q(r - \| X_i - X_j \|), \qquad (4)$$

where r is the radius, m is the embedding dimension, X_i and X_j are m-dimensional histories, and $q(s)$ is Heavyside function that is given by $q(s) = 1$ if $s \geq 0$, and $q(s) = 0$ if $s < 0$. The correlation integral provides the fraction of pairs (X_i, X_j) having distance between them smaller than a chosen radius *r*, and may be interpreted as the probability of the distance between any two *m*-histories being smaller than *r*. GP show that $C(r) \propto r^{D_c}$, and define the correlation dimension as:

[16] See for example, Diks (1999, p.19).

$$D_c^m = \lim_{M\to\infty}\lim_{r\to 0}\frac{\log C(M,m,r)}{\log r}. \quad (5)$$

The correlation dimension may be considered as an effective number of degrees of freedom or "a lower bound on the number of independent variables which would be required to model the series" (Brooks, 1998, p.272).

When analyzing empirical data, the number of *m*-histories that can be created, *M*, is finite, and correspondingly the radius *r* cannot be too small. In order to distinguish between deterministic chaos and random noise, one embeds data in higher and higher spaces (by increasing the embedding dimension *m*) and calculates the slope of *log C(M,m,r)* against *log(r)* for each *m*. For random noise this slope will increase indefinitely with the increase in *m* (slope will be approximately equal to *m*). For a signal that comes from a deterministic system the slope will reach the value of correlation dimension and then will become independent of further increases in *m*.

Very important modification to GP method was proposed by Theiler (1986)[17]. He showed that in the limited data sets correlation dimension estimates calculated directly from the GP formula would be biased downward due to the "temporal" correlation in the data. Temporal correlation, as opposed to geometrical correlation (that the correlation integral is supposed to measure), is the outcome of the fact that points on the reconstructed orbit that are close in time, are also close in the phase space. If those points are not excluded when calculating the correlation integral, estimates would be understated and it is possible to obtain low estimates even for stochastic systems. In order to overcome this problem, the GP formula has to be modified as follows:

$$C(M,m,r) = \frac{2}{(M-W)(M-1-W)}\sum_{i=1}^{M-W}\sum_{j=i+W}^{M} q(r - \| X_i - X_j \|), \quad (6)$$

where $W \geq 1$ is the Theiler correction (Diks, 1999, p.20) which is recommended to be chosen at least as high as the autocorrelation time. We believe that this correction is not frequently used in economic empirical studies and this might be the reason for a number of small estimates reported that are just artifacts of the dynamical correlation in the data.

GP (1983b) also notes that if there is some random noise that is imposed on an otherwise deterministic chaotic signal, then a plot of *log C(M,m,r)* on *log(r)* will have two

[17] See Kantz and Scheiber (2002, p.80) for the discussion.

regions. In the region with the small *r*, in which random noise will dominate chaotic signal, the slope of the line will be approximately equal to the embedding dimension. In the region with the higher *r*, the slope of the *log C(M,m,r)* against *log(r)* will converge to the correlation dimension of the deterministic attractor. Eckmann and Ruelle (1985) call such plots "curves with knees", and their presence may indicate that an underlying chaotic signal is contaminated with noise.

One of the limitations in the use of this indirect classification test for chaos is that the number of data points in empirical applications is usually very small while the precise calculation requires an almost infinite amount of data. Eckmann and Ruelle (1992) provide the formula that allows calculating an approximate maximal possible estimate of the correlation dimension:

$$d_{\max} = \frac{2\log N}{\log(1/r)} \tag{7}$$

where $d_{\max}$ is the maximal possible estimate of correlation dimension for a given data set; N is the number of data points in the scalar time series;

and $r = r/D$ is relative magnitude of *r* used in the GP algorithm in relation to the diameter of the reconstructed attractor. As the GP algorithms calls for *r* that approaches zero, Eckmann and Ruelle (1992) argue that it is necessary to look at $r \leq 0.1$. The paper concludes that a reliable estimate of dimension has to be substantially below $d_{\max}$, and it can be calculated only when the linearity of GP plots at small *log r* can be verified, and there is convergence of slopes for different embedding dimensions.

Another limitation in the use of correlation dimension as the chaos classifying tool is the absence of a theory that would provide its distributional characteristics[18]. Therefore, in this paper, we are interested not as much in the exact estimates of the dimension for different quantities, but in the qualitative picture of the convergence or divergence of those estimates, and also in the similarities or differences in the estimates for different currency pairs.

[18] See, for example, Barnett et al. (1995).

2.4 **Maximum Lyapunov exponent**

A distinguishing feature of chaotic dynamical systems is SDIC. SDIC makes deterministic chaotic systems completely unpredictable in the long run because the distance between two orbits that start from very close initial points grows exponentially. Lyapunov characteristic exponents quantitatively measure the stability or instability of the dynamical system. A necessary condition for the presence of chaotic dynamics in a time series is a positive maximal Liapunov exponent.

One of the first methods developed to estimate the maximum Lyapunov exponent from data was by Wolf et al (1985). However, it requires a large amount of data and is sensitive to noise. In this paper we are using the Rosenstein et al (1993) method which is simple compared to other methods[19], and is tested to work well in small samples and with data contaminated with noise.

This approach relies on the method of delays for the phase space reconstruction, similar to the GP algorithm for correlation dimension. After the dynamics of the system is reconstructed and orbit $X=(X_1, X_2, ..., X_M)'$ obtained, the next step is to locate the nearest neighbor for each point, and calculate the initial distance between nearest neighbors:

$$d_j(0)=\min_{X_i} \| X_j - X_i \|_p , \tag{8}$$

where $\|...\|_p$ denotes the norm in m-dimensional reconstructed space[20]. The search for the nearest neighbor is constrained by the requirement that nearest neighbors should have a temporal separation larger than the mean period of the time series, which can be calculated as the reciprocal of the mean frequency of the power spectrum. This requirement allows interpreting the nearest neighbors as two close points from different trajectories. The largest Lyapunov exponent may be estimated as the mean rate of separation of nearest neighbors.

Rosenstein et al (1993) method then proceeds as follows. As a maximum Lyapunov exponent measures the speed of exponential divergence or convergence, the

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[19] It was used by Bask (2002) and Brzozowska and Orlowski (2004) in the context of exchange rate time series.

[20] The Euclidean norm is proposed in the original paper but any other norm will work. We will use the "maximum" norm in our estimation.

following formula may be used to show the relation connecting initial distance between nearest neighbors and distance after some time, *t*:

$$d_j(t) \approx d_j(0) * \exp(l_1 * t), \tag{9}$$

where *t* is the number of separation steps, l_1 is the largest Lyapunov exponent, and $d_j(t)$ is the distance between nearest neighbors after *t* separation steps.

By taking the logs of both sides of the above formula we obtain:

$$\log d_j(t) \approx \log d_j(0) + l_1 * t \tag{10}$$

Then averaging over all the pairs of nearest neighbors (over all *j*), we get

$$\langle \log d_j(t) \rangle \approx \langle \log d_j(0) \rangle + l_1 * t \tag{11}$$

where $\langle \ldots \rangle$ denotes the average over all values of *j*. Then the largest Lyapunov exponent, l_1, is estimated by regressing the average log distance after *t* separation steps on the number of separation steps.

Rosenstein et al. (1993) shows that this method works satisfactorily even for very small data sets, is not very sensitive to time delay, and provides good estimates even when some small amount of observational noise is present (1-10%). What is more important, it allows discriminating between chaotic and stochastic systems. In stochastic systems nearest neighbors will neither diverge nor converge, and a flat plot of $\langle \log d_j(t) \rangle$ versus *t* is expected.

3. Previous empirical studies of chaos in exchange rates

The interest in the possibility of chaotic dynamics in economic time series emerged in the 1980s, and it was stimulated by the theoretical possibility of explaining seemingly random fluctuations and large movements in financial markets by means of deterministic systems[21] (Hsieh, 1991). Major methodologies and testing procedures were established at that time, mostly by adjusting the tools used in physics and other natural sciences. A summary of the results of the initial empirical studies is reported in LeBaron (1994). The main conclusion of the early literature is that empirical evidence for chaotic

[21] This implies that chaotic models are able to generate nonlinear dynamics endogenously while the traditional models rely on the external shocks to have interesting dynamics.

dynamics in economic time series is very fragile, although most studies found some support for the presence of nonlinear structures in the examined series. Hsieh (1991) stresses that only low dimension chaotic behavior is of interest to economists, as high dimensional chaotic processes are difficult to detect with limited amount of data.

Studies dedicated to the search for chaos in asset prices, and in exchange rates in particular, may be subdivided into positive, cautiously positive and negative by their results concerning chaos in the data. The methodological tools also differ from paper to paper.

One of the early indirect studies of chaos in exchange rates is by Hsieh (1989). Hsieh researches whether exchange rate returns on five major currencies contain any nonlinearities. As the presence of nonlinearity is the important prerequisite for chaos, such study is the necessary first step. The paper applies the BDS test, which strongly rejects the iid hypothesis in all currency pairs, even when linear dependence is filtered out by an autoregressive process. Trying to discriminate between stochastic and deterministic nonlinear dependence, Hsieh fits the AR-GARCH (1, 1) model to the returns and then applies the BDS test to the standardized residuals. While stochastic origin is confirmed for Canadian Dollar, Japanese Yen and Swiss Franc returns, strong unexplained nonlinearity is still present in British Pound returns and in returns of the Deutsche Mark, leaving question of the origin of nonlinearity in those currencies open.

Among the early cautiously positive studies of empirical chaos is the paper by Scheinkman and Lebaron (1989), which considers daily and weekly returns on the value-weighted stock portfolio. They report correlation dimension equal to 5.7 for weekly returns, which does not imply low-dimensionality of the studied series but suggests some kind of nonlinear structure in the data. This result is strongly reinforced by the experiment with the “scrambled” series that are obtained by regressing original return data on their past values, estimating residuals, and then adding the randomly selected residuals to the estimated linear system. Estimates of correlation dimension for scrambled data turned out to be much higher than for the original series.

De Grauwe et al. (1993) provide mixed evidence on chaos in exchange rates. They consider the period from 1973 to 1990 and their sub-periods and calculate the dimension for DM/USD, BP/USD and JPY/USD returns. Absence of chaos is reported for DM/USD, but rather strong indication of chaos, with the estimates of correlation dimension around 2, are obtained for BP/USD and JPY/USD for 1973-1982 and the overall period.

A number of chaos positive papers appear in the literature. The study by Bajo-Rubio et al. (1992) applies the correlation dimension and the Lyapunov exponents to detect the presence of chaos in Spanish Peseta – U.S. dollar spot and forward rates, using daily data for 1985-1991. Due to the presence of unit root in the level series, tests are applied to first differences of the original data. The correlation dimension found ranges from 2.7 to 3.2 for spot rates, and 1.8 to 3.3 for forward rates, suggesting the presence of low dimensional chaotic dynamics. Lyapunov exponents, calculated using the Wolf algorithm, are all positive, further supporting the presence of chaos.

Bask M. (2002) studies the chaos in daily Swedish Krona exchange rates against a few major currencies using the Rosenstein et al (1993) method to calculate the maximum Lyapunov exponent and the blockwise bootstrap procedure of Bask and Gencay (1998) to find the critical values of the estimates. His estimates are positive in many sub-periods, compatible with the presence of chaos. The same methodology is applied by Brzozowska-Rup and Orlowski (2004) to daily average US Dollar against Polish Zloty exchange rates for the period 1993-2003, and also for some embedding dimensions estimates of the maximum Liapunov exponent are positive, which is consistent with chaos.

A few papers report a definite rejection of chaos in exchange rates. The paper by Brooks (1998) studies ten daily sterling-denominated exchange rate returns. A few of the series exhibit some weak saturation of the correlation dimension estimates at around 4.5-6 level when the embedding dimension is increased to 15. Then the Wolf et al. (1985) algorithm and the Dechert and Gencay (1990) neural network technique are applied to determine the Lyapunov exponents. Estimates produced by the Wolf algorithm are positive in all cases, but this is invalidated by similar positive estimates obtained for surrogate data. Estimates obtained from the second method are consistently negative and the overall conclusion reached is strictly chaos negative.

Guillaume (1995, 2000) uses intraday exchange rates to study the possibility of chaos in exchange rate returns and their absolute values using the GP (1983) algorithm. The author looks at different sampling frequencies, namely 15, 30, and 60 minutes, over the 6-year period starting at the beginning of 1987 and until the end of 1992 for four major currency pairs: USD/DEM, USD/GBP, USD/JPY, and USD/FRF[22]. A few different methods of the attractor reconstruction are used. Neither of the methods can produce the convergence of the correlation dimension estimates suggesting the absence of low dimensional chaos in the studied series.

Serletis and Shahmoradi (2004) provide convincing evidence of the absence of chaotic dynamics in CAD/USD returns for the 1974-2002 time period. They calculate the maximum Lyapunov exponent using the Nychka et al. (1992) method and additionally construct the confidence intervals for the estimates using the procedure presented in Shintani and Linton (2004). Estimates obtained for the embedding dimension up to 6 are all negative, rejecting the low dimensional determinism in this exchange rate.

4. Data description

In order to test for evidence of chaotic dynamics in exchange rates, we consider 4 major currency pairs, namely U.S. $ to 1 British Pound (USD/BP), Japanese Yen to 1 U.S. $ (JPY/USD), Swiss Franc to 1 U.S. $ (CHF/USD), and Canadian $ to 1 U.S. $ (CAD/USD), given by noon buying rates in New York and reported at the webpage of the Federal Reserve Bank of St. Louis[23]. Our data set spans the period from January 1975 until June 2006 and contains 7897 daily observations for each currency pair.

Similarly to many other papers, we study the exchange rate returns, but additionally we include volatility and normalized exchange rates, definitions of which are provided below. Let S_t denote the level of the exchange rate at time t. *Exchange rate return* at time t is calculated as the log difference of consecutive exchange rate levels:

$$r_t = \ln(S_t) - \ln(S_{t-1})$$

We define daily *volatility* as the absolute value of exchange rate returns:

$$V_t = |r_t|.$$

[22] A small number of data points is a typical drawback of the empirical studies of chaos in economics, and the use of intraday data seemingly overcomes this problem. But, as is mentioned by Hsieh (1991), high frequency data are contaminated by artificial dependencies caused by the market microstructure effects of forex trading. At the same time when long time series over many years are used, then there is problem of nonstationarity.

[23] http://research.stlouisfed.org/fred2/categories/94

Normalized exchange rate is created as current exchange rate divided by a 50-day simple moving average:

$$X_t = S_t / \sum_{i=0}^{49} S_{t-i} \, ,$$

and it may be regarded as a technical trading signal[24]. As similar technical signals are heavily used in the forecasting by technical traders we are curious to see if any deterministic dynamics can be found in such variable.

Figures 1-4 present a graphical description of the above mentioned series; and tables 1 (i-iv) show summary statistics. The same series, with the addition of DEM/USD were studied in Hsieh (1989), for the shorter period of 1974-1983. We would like to note that the returns in our sample have smaller kurtosis than in Hsieh's sample, implying closer approximation by the normal distribution, although fat tails are still characteristic for these series. While the mean level of the returns is close to zero, there are instances when daily changes in exchange rates are high, like -5.65% for JPY/USD or 5.8% for CHF/USD. The least volatile exchange rate in our sample is CAD/USD, with lowest volatility and smallest range of returns. The highest volatility is observed in the CHF/USD currency pair.

An important requirement for the time series analysis is the stationarity of the data. Augmented Dickey-Fuller and Phillips-Perron unit root tests are applied to check for the stochastic trend in the exchange rate levels. All specifications suggest a presence of the unit root in the level series[25]. Therefore in what follows we will study first differences of exchange rates (exchange rate returns) that are stationary according to the above mentioned tests. Normalized exchange rates and volatilities also pass the tests for stationarity.

Figure 5 provides the description of the serial correlation in our data. Exchange rate levels are highly persistent, with first zero of autocorrelation function ranging from 781 lags for USD/BP to 2574 lags for CHF/USD. Log-differencing removes almost all autocorrelation, and as a result exchange rate returns are practically uncorrelated[26]. At the

[24] I thank Blake LeBaron for suggesting me to consider this variable.
[25] Except the JPY/USD exchange rate, for which the null hypothesis of unit root cannot be rejected at the 1% significance level, but is rejected at the 5% significance level.
[26] Autocorrelation is considered significant with 95% confidence if it lies outside the interval [$-2n^{-1/2}$, $2n^{-1/2}$], where *n* is the number of data points used in calculation.

same time, absolute values of returns (volatility variables) exhibit strong linear dependence which differs substantially among currencies[27]. Normalized exchange rates are much less persistent than exchange rate levels, and their autocorrelation declines to zero after approximately 40-100 lags, but if the number of legs is increased further, prolonged periods of nonzero autocorrelations are present, reflecting the dependence structure that is imposed on this variable by including the 50-day moving average.

For the low-dimensional chaotic systems it is possible to provide a qualitative information about the dynamics of the system by means of a phase portrait. It is constructed as a plot of two or more dynamical variables against each other. For experimental data a phase portrait is constructed by plotting the variable of interest against the same variable but delayed in time. The time delay can be chosen similarly to the phase space reconstruction as the first zero of ACF, first minimum of AMI or using some other method.

Figure 6 provides phase portraits for all variables in two-dimensional space[28]. Time delays are chosen to be equal to the first zero of the ACF[29]. Plots for exchange rate levels look quite interesting and suggest that there is some nonlinear dependence structure present there, implying the possibility of chaotic dynamics. But non-stationarity and very large delays required for the elimination of the autocorrelation prevents us from applying chaos tests directly to exchange rate levels. Plots for returns illuminate a quite different picture: points on the reconstructed trajectory are distributed much more evenly in the phase space, with the concentration in the region close to zero. Even if there is some structure in these series it would require a much higher-dimensional space to see it. Phase portraits for normalized exchange rates are visually similar to the plots for returns series, but they exhibit more structure at the same time. And graphs for volatilities, that are the projections of returns' plots onto the first quadrant, do not suggest the presence of a low dimensional strange attractor.

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[27] The first zero of ACF for CAD/USD volatility is observed after 1631 lags, for JPY/USD volatility – after 445 lags, for CHF/USD volatility – after 90 lags, and finally for USD/BP volatility – after 324 lags.

[28] Certainly if the dimension of the attractor is much higher than 2 then phase portrait in two dimensional space will be not very informative.

[29] For exchange rate levels they are as following: CAD/USD – 1588, JPY/USD - 2555, CHF/USD - 2573, and USD/BP – 780. As returns are practically uncorrelated we choose time delay 1 for them. Delays for normalized exchange rates: CAD/USD – 43, JPY/USD – 88, CHF/USD – 73, and USD/BP – 105. And finally delays for volatilities are as follows: CAD/USD – 1663, JPY/USD – 444, CHF/USD – 197, and USD/BP – 323.

Before proceeding to the calculation of invariants of motion we first apply the BDS test to the return series in order to establish the presence of nonlinearity in the data. Following Hsieh (1989), we define distance *r* in terms of the standard deviation of the data, ranging from 0.5 to 1.5, while setting the maximum embedding dimension *m* to 10.

Table 2 (a) reports the BDS statistics for exchange rate returns[30]. The null hypothesis of iid is strongly rejected for all currency pairs. The highest degree of dependence is observed in CAD/USD returns for which the BDS statistics is consistently higher in all cases, which is then followed by USD/BP, JPY/USD and finally CHF/USD returns. Rejection of the null hypothesis in our sample is slightly stronger than in Brock et al (1991)[31], where the same exchange rates but for a much shorter period (1974-1983) are considered.

In order to remove possible linear dependence in the data we filter returns with the autoregressive model with maximum number of lags set to 10. We use Akaike Information Criterions (AIC thereafter) for the selection of the best model. Those are AR(0) for all returns except USD/BP, for which AR(1) was selected. The resulting BDS statistics for filtered returns are very close to the ones for the raw data. That confirms that the rejection of iid in return data is due to some nonlinear dependence in the series.

The above analysis suggests that our data is generated by some nonlinear processes, which can be stochastic or deterministic. A number of nonlinear stochastic processes are described in the time series literature and it is impossible to test each of them. Many empirical financial publications suggest that GARCH type models provide an accurate description of the return data. Therefore, we fit different AR(p)-GARCH(m,n) models to the returns[32] and then apply the BDS test to the standardized residuals of the models selected by the AIC criteria.

AIC selects the following models: CAD/USD – AR(2)-GARCH(2,2); JPY/USD – AR(1)-GARCH(3,1); CHF/USD – AR(1)-GARCH(1,1); and USD/BP – AR(9)-GARCH(3,1). The results of the BDS test on the standardized innovations of these models are reported in table 2(b). We use the quantiles of the standard normal distribution to

[30] For the calculation of BDS statistic we used a Matlab program provided by Blake LeBaron which is available at http://people.brandeis.edu/~blebaron/soft.html.
[31] See table 4.3, page 148.
[32] Maximum *p* is set to 10 and maximum *m* and *n* are set to 3.

establish the significance of the estimates, although as was noted by Hsieh (1989)[33] among others, use of the standard normal distribution for GARCH standardized residuals is rather conservative, and critical values have to be lower in absolute value.

The BDS test finds slight nonlinearity for CAD/USD and CHF/USD at high embedding dimensions, but very strong nonlinearity for JPY/USD and USD/BP, except cases when dimension, *m*, is low (2, 3, 4) and distance, *r*, is large (1.25 or 1.5). Therefore, we may conclude that while CAD/USD and CHF/USD are accurately described by the stochastic AR-GARCH model, the question of the best model for USD/BP and JPY/USD returns remains open, and those two currency pairs are potential candidates for representation by some deterministic model. We will continue this investigation in the next section by applying correlation dimension and maximum Liapunov exponent criteria to these series.

5. Empirical results

5.1 Correlation dimension estimation

Correlation dimension is estimated using the GP algorithm outlined in section 2.3. In all cases we calculate the correlation integral using the "maximum" norm[34] that measures the distance between two vectors as the maximum of the differences between corresponding coordinates of those vectors. Embedding dimension is set in the range from 2 to 12. Time delay, t, is chosen according to the formula: $ACF(t) \leq (1-1/e)ACF(1)$, where ACF is the autocorrelation function[35]. Given that we have 7896 data points for exchange rate returns, the rough[36] estimate of $d_{\max}$ for our data set is 7.8, calculated using the Eckmann and Ruelle formula reported in section 2.3.

[33] See Hsieh (1989), page 363, especially note 4 on that page. See also Brock et al (1991), Appendix F, that provides distribution and quantiles of the BDS statistic GARCH (1,1) standardized residuals for different number of observations and different distance parameter.
[34] Use of the Euclidean norm does not change results qualitatively but makes it more difficult to establish scaling regions.
[35] This method of choosing time delay was reported to provide the best performance of Rosenstein et al (1993) algorithm for maximum Lyapunov estimation that we use in this paper.
[36] Assuming that the diameter of the attractor at the higher embedding dimensions is approximately equal to the diameter at dimension 1.

And finally we consider radius in the range from 0.1 to 2.5 standard deviations of the data with the step equal to 0.05 standard deviations.

In order to evaluate the accuracy of our Matlab code we first test it on the data produced by the Henon map, for which the true estimate of correlation dimension is known and is approximately equal to 1.21-1.25[37]. The Henon map is given by the following system of difference equations:

$$x_{t+1} = y_t + 1 - 1.4 * x_t^2,$$
$$y_{t+1} = 0.3 * x_t.$$

Using (0,0) as the initial condition, the orbit of the Henon map is characterized by chaotic dynamics whose strange attractor is shown in figure 7. We generate 8896 data points by the above system, discard first 1000 points as transients, and then use the last 7896 of the *x*-series for the estimation of the dimension in order to see how the algorithm works on a sample size comparable to ours.

The resulting plot of log correlation dimension against log radius for embedding dimensions from 2 to 12 is presented in figure 8. A clear scaling region is obvious for small values of *r*, on which the slope is linear and similar for different embedding dimensions (this region corresponds roughly to the range of radius from 2.8% to 7% of the attractor's diameter). In the region further to the right of this range (the furthest right value of the radius is equal to 70% of the attractor) scaling behavior is absent, and it would not be possible to calculate a correlation dimension in this region. Hence, "small" value of radius is an important prerequisite for obtaining good estimates. Numerical estimates in table 3 show that the algorithm provides a very accurate approximation of the correlation dimension for the actual chaotic data[38].

5.1.1 Exchange rate returns

Correlation dimension for exchange rate returns is obtained through the following steps. First we apply the GP algorithm with Thailer correction, *W* set to 100, to the raw

[37] Grassberger and Procaccia (1983b, p201).
[38] We still have to be cautious when applying this algorithm to the actual data, because as was mentioned by Kantz and Schreiber (2004, p.78), although correlation dimension estimation is a good tool to measure self-similarity (that can be inferred from fractal dimension) in known chaotic systems, it is less appropriate when applied to systems, the chaotic origin of which has yet to be established.

return series. Then we recalculate the dimension for scrambled returns, following the procedure suggested by Scheinkman, J. & LeBaron (1989) with small modification[39]. The idea behind this procedure is that if our original data has some deterministic structure, the reshuffling should substantially destroy such dependences, and the system should be more “random” than the original one. As a result, obtained estimates of correlation dimension should be higher for the reshuffled data.

Similar steps are then applied to the standardized residuals of AR-GARCH models fitted to return series. As was shown in section 4 by the BDS test, strong nonlinear dependence is still present in the residuals of JPY/USD and USD/BP return series, and we expect to see it in the GP plots.

Results are presented in figures 9 (a-d), and table 4 reports estimates of correlation dimension for cases where a clear scaling region is present.

Opposite to the Henon map, the same method of choosing the radius for return series (0.1:0.05:2.5 of standard deviation of the data) results in a relative range between 1% and 24% of the attractor’s diameter[40].

The plot for CAD/USD returns (figure 9(a)) indicates a stochastic origin of the series, as expected. This is visible from the absence of any scaling region in the plot of local slope of the log correlation integral against log of radius, and an increase in the estimates of correlation dimension with each subsequent embedding dimension. The plot for CAD/USD scrambled returns looks very similar, with the only difference being that for high embedding dimensions and small values of radius there are no sufficient data points and the plot looks jagged in that region. The same absence of evidence for chaos is also apparent in the graph for standardized GARCH residuals of CAD/USD.

Results for JPY/USD are noticeably different. First of all, there is a clear scaling region for the middle range of radius (it is shown by dotted lines in fig.9 (b)), which corresponds to the region between 2.9% and 7.48% of the diameter of attractor. The slope shows slow tendency to saturation at around 5.3, but saturation is not complete and a small increase in correlation dimension is still observed when the embedding dimension

[39] Scheinkman and LeBaron (1989) regress the original data on past data and estimate residuals from those models. Then they sample with the replacement from the residuals and use the obtained sample for rebuilding the original data using the estimated linear models and the same initial values. We sample with replacement directly from the return series, and consider it the “scrambled” returns.

[40] This is the other way to see that return series have substantial probability of extreme values as compared to the Henon map.

goes up. At the same time estimates for scrambled returns are substantially higher, pointing out that some structure is present in the return series, suggesting the possibility of chaos. But applying formula (7) for the range of radius that corresponds to the scaling region, it is revealed that the maximum possible reasonable estimate of dimension in that range is 5 – 6.9. Therefore, our estimate of 5-6 is too close to the critical region, and we conclude that such a result is due to the small number of data points rather than finite dimension of the attractor. This conclusion is reinforced by the graph for AR-GARCH standardized residuals. When JPY/USD returns are adjusted for AR-GARCH structure, the saturation of the slope disappears and series behaves as purely random.

The case of CHF/USD returns is definitely of stochastic origin. No saturation of the slope is observed, and the qualitative picture is similar for pure and scrambled returns, especially when AR-GARCH structure is removed (see fig.9 (c)).

Fig.9 (d) shows the results for USD/BP returns. Although there is a jump in the dimension estimate when data is scrambled; there is no definite scaling region for raw returns to calculate the dimension. But when the AR-GARCH effect is taken into account, a comparatively good scaling region, ranging from 1.3% to 4.7% of diameter, is observed, on which we may estimate correlation dimension as approximately 5.9 (table 4). However formula (7) indicates that the maximum possible dimension in this region is between 4.20 and 5.86. Hence, our conclusion is chaos negative in this case.

In summary, we may conclude that there is heterogeneity in terms of data generating systems for exchange rate returns. For CAD/USD and CHF/USD returns, a conclusion of stochastic origin seems unquestionable, and that is what we would expect given the results of the BDS test. At the same time, JPY/USD and USD/BP returns have some structure with possible dimensions between 5 and 6. As was shown by Eckmann and Ruelle (1992), with a given number of data points these estimates cannot be interpreted as indication of finite dimension of the attractors. We think that the last two cases may be interpreted as weak support for some finite-dimensional deterministic system with dynamical noise.

5.1.2 Normalized exchange rates

Normalized exchange rates by construction have very persistent autocorrelation structure (AIC suggests that the number of lags that has to be included in the AR models has to be higher than 50). Although this high autocorrelation is partly accounted for when reconstructing the phase space by including high time delays, the points on the reconstructed trajectory are still temporarily correlated. That suggests that using Theiler correction for the GP algorithm is critically important in this case. Omitting this adjustment would lead to a wrong conclusion of low-dimensional attractor as is shown below.

Figure 10 (a) reports the estimation of correlation dimension for the CAD/USD normalized exchange rate. The Theiler adjusted GP algorithm displays stochastic character of the normalized variable as no linear scaling region or saturation is present. The lower panel of the figure also shows the estimation when correction for temporal correlation is not done. For small values of radius there is a scaling region with constant slope that changes little with the increase in embedding dimension. Numerical estimation would incorrectly suggest the correlation dimension between 3.5 and 4.5, which can be mistakenly taken as strong evidence of chaos in this series.

Figures 10 (b-d) show that the same conclusion applies to all other normalized exchange rates, namely all normalized returns are the product of some stochastic system. Failure to account for temporal correlation would result in a wrong conclusion of low dimensional chaos in the series with minimum number of degrees of freedom around 3-4. It is interesting that the same behavior can be observed for an artificially constructed series of a normalized random walk.

5.1.3 Volatility

Analysis of the volatility series shown in figures 11 (a-d) indicates that they are generated by stochastic systems as well. Once again the importance of Theiler correction is underlined, as dimension estimates rise substantially when temporal correlation is excluded.

The overall conclusion from the correlation dimension estimation exercise is that all the series under consideration can be classified as stochastic. There is some weak indication of some nonlinear finite-dimensional structure for JPY/USD and USD/BP returns that could indicate the presence of deterministic origin coupled with dynamic noise but this cannot be established with the given number of data points. In the next section we will refer to the maximum Lyapunov exponent estimation as an ultimate test for chaos.

5.2 Results of Rosenstein method for maximum Lyapunov exponent

Rosenstein et al (1993) suggests that it is possible to distinguish non-chaotic stochastic systems from chaotic ones by the non-linear scaling region due to the divergence of nearest neighbors that is not exponential and flattening of the scaling region when embedding dimension is increased. Also for stochastic systems the graph of the average log distance against the number of separation steps is expected to be almost flat, implying that on average there is neither convergence nor divergence between nearest neighbors. Stochastic systems may be also characterized by an initial jump at initial separation steps.

For a numerical implementation of the algorithm we use the same parameters set that was used for the correlation dimension calculation. The minimum time separation is set to 100, and number of steps over which we try to establish the maximum Lyapunov exponent is set to 20. According to Rosenstein et al (1993) the proper number of separation steps corresponds to the region at which the graph of the logarithm of distances between nearest neighbors against the number of separation steps has constant slope.

Data generated by the Henon map is used to test the performance of Matlab implementation of the Rosenstein method. Results of this test are presented in figure 12 and table 5. As is evident from the graph, there is a sufficiently large scaling region between steps 3 and 7 that allows very accurate estimation of MLE for the Henon map, which is equal to 0.418[41]. It is best seen in the graph of the local slopes against the number of separation steps. In the scaling region the slopes are horizontal. The horizontal region becomes smaller and smaller with each additional embedding dimension, and

[41] See, for example, Wolf et all (1985).

correspondingly the precision of estimates slightly deteriorates. As is argued by Rosenstein et al (1993), satisfactory results are obtained when *m* is at least equal to the topological dimension of the system (it is approximately equal to 1.22 for Henon map) and remains accurate until it is below the Takens criteria for embedding dimension. Estimates for even higher embedding dimensions would loose precision. The numerical estimates are reported in table 5. As can be seen from the table, the method is very accurate for low-dimensional chaotic system.

5.2.1 Results of Rosenstein method for exchange rate data

Plots of MLE for exchange rate returns are presented in figure 13[42]. The only evident horizontal line is that for the slope of zero, for all return series, raw and scrambled. The plots differ only in how the zero slope is approached. The exact shape of the slope curves depends on the number of delays used in the phase space reconstruction. For example, for the raw return series the time delay was set to 2, for AR-GARCH residuals of JPY/USD the delay is 6, and for the scrambled data delay is 1. Graphs for the scrambled series are the exact replica of what was obtained by Rosenstein et al (1993) (fig.9, page 130) for white noise. Therefore, this graphical evidence suggests that all the return series are some kind of random processes as all the prerequisite are met, such as initial jump, nonlinear scaling region, flattening of the scaling region with increasing embedding dimension.

Quite similar results are obtained for normalized exchange rates and volatility series (fig. 14). Plots either exhibit dying oscillations around zero for raw data, or slow flattening of the slope with increasing embedding dimension for scrambled series[43].

The main conclusion from this exercise is that there is no evidence of sensitive dependence on the initial conditions in any of the variables considered. The hypothesis of chaotic dynamics in exchange rates is strongly rejected.

[42] We present only JPY/USD case as all other cases are qualitatively identical.

[43] Time delays used in the phase space reconstruction are equal to 15 for JPY/USD normalized ER, 12 for volatility and 1 for scrambled data. In all cases temporal separation between nearest neighbors is set to 100.

6. Conclusions

In this paper we evaluate the evidence for chaotic dynamics in three variables derived from exchange rates, namely, exchange rate returns, volatility and normalized exchange rates. Chaotic dynamics can be characterized by a number of features that has to be observed in the data, and we concentrate on three of them: nonlinearity, low dimensional attractor and sensitive dependence on initial conditions. All three characteristics are required for chaos identification.

Our results suggest that exchange rates exhibit strong nonlinear dependence which is not completely accounted for by GARCH-type models, and that different nonlinear empirical models are required to fit the data for different currencies.

The possibility of pure chaotic dynamics in empirical data is strongly rejected by both major chaos-identifying tools – correlation dimension and maximum Lyapunov exponent.

We found that the correlation dimension of the signals extracted from the exchange rates keeps increasing with the increase in the dimension in which the dynamics is embedded. This is a clear indication of non-chaotic origin of the series. At the same time we showed that if the GP algorithm for dimension calculation is applied directly, without the adjustment for temporal correlation of the points on reconstructed orbit, it could spuriously generate low estimates of the dimension. We believe that at least some of the published results of low dimensional chaos in financial data suffer from this drawback.

The graphical procedure for calculation of maximum Lyapunov exponent calculation suggested by Rosenstein et al (1993) reveals the absence of sensitive dependence on initial conditions.

In order to make any strong conclusion about the possibility of low dimensional chaos in exchange rate data it is necessary to address the question of stochastic noise. We believe that a small amount of observational noise would not result in the rejection of the chaos hypothesis by the tests we use. But if a small amount of dynamical noise, which is amplified by the underlying nonlinear system, is present then chaos identification tools would not be able to recognize the chaotic skeleton of the model. That is illuminated by Hommes and Manzan (2006) who use the chaotic asset pricing model proposed by Brock and Hommes (1998) that is subject to a small amount of dynamical noise. They show that when the inverse signal-to-noise ratio is increased to 0.12, the estimate of the maximum Lyapunov exponent for otherwise low dimensional chaotic model becomes almost zero. Further increase in the level of noise results in negative maximum Lyapunov exponent and, hence, rejection of the underlying chaotic model.

Therefore, while the question of the possibility to detect empirically low-dimensional chaotic signals contaminated with dynamic noise remains open, we would like to suggest that dynamical noise has to be included in theoretical models of asset price determination if they are intended to generate series comparable with empirical data.

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Appendix

Table 1 (i): Summary statistics for daily exchange rates, January 1975 - June 2006

	CAD/USD	JPY/USD	CHF/USD	USD/BP
Mean	1.2817	165.5438	1.6898	1.7023
Median	1.2640	133.4500	1.5580	1.6535
Std Dev	0.1486	63.1410	0.4123	0.2478
Kurtosis	2.4216	2.1195	2.8658	3.8011
Skewness	0.1065	0.7524	0.9387	0.8244
Min	0.9628	81.1200	1.1172	1.0520
Max	1.6128	306.8400	2.9245	2.4460

Table 1 (ii): Summary statistics for normalized exchange rates, January 1975 - June 2006

	CAD/USD	JPY/USD	CHF/USD	USD/BP
Mean	1.0004	0.9973	0.9981	0.9994
Median	1.0004	0.9996	0.9997	0.9997
Std Dev	0.0129	0.0299	0.0315	0.0270
Kurtosis	4.3185	4.1075	3.1222	4.5648
Skewness	-0.1588	-0.4425	-0.1227	-0.1752
Min	0.9309	0.8460	0.8754	0.8623
Max	1.0588	1.1163	1.1083	1.1298

Note: Normalized exchange rate is the index number, that is equal to 1 when current ER is equal to 50-days moving average. It is larger than one when current ER is higher than 50-day moving average.

Table 1 (iii): Summary statistics for exchange rate returns, January 1975 - June 2006

	CAD/USD	JPY/USD	CHF/USD	USD/BP
Mean	0.0013	-0.0123	-0.0093	-0.0030
Median	0	0	0	0.0056
Std Dev	0.3195	0.6553	0.7332	0.6078
Kurtosis	6.0478	7.3341	5.8310	6.5563
Skewness	0.0278	-0.4922	-0.0190	-0.1320
Min	-1.8642	-5.6302	-4.4083	-3.8427
Max	1.9029	3.5571	5.8269	4.5885

Note: Exchange rate returns are calculated by taking the logarithmic differences between successive trading days

Table 1 (iv): Summary statistics for exchange rate volatility, January 1975 - June 2006

	CAD/USD	JPY/USD	CHF/USD	USD/BP
Mean	0.2276	0.4606	0.5381	0.4326
Median	0.1617	0.3253	0.3967	0.3083
Std Dev	0.2242	0.4663	0.4982	0.4270
Kurtosis	9.1629	12.7140	10.4344	10.9676
Skewness	2.1006	2.3385	1.9859	2.1606
Min	0	0	0	0
Max	1.9029	5.6302	5.8269	4.5885

Note: In this paper exchange rate volatility is calculated as absolute values of daily exchange rate return.

Table 2 (a). BDS test: exchange rate returns, raw data

m	*r*	CAD/USD	JPY/USD	CHF/USD	USD/BP
2	1.5	19.846	11.687	8.9955	13.142
3	1.5	24.651	14.153	11.42	16.43
4	1.5	28.348	16.373	13.722	18.581
5	1.5	31.777	18.208	15.682	20.505
6	1.5	35.238	19.882	17.603	22.816
7	1.5	38.706	21.507	19.469	24.981
8	1.5	42.439	23.012	21.151	26.913
9	1.5	46.451	24.724	22.985	28.936
10	1.5	50.673	26.558	24.872	31.23
2	1.25	20.158	11.767	8.6385	13.479
3	1.25	25.268	14.478	11.253	17.101
4	1.25	29.514	17.129	13.766	19.614
5	1.25	33.686	19.576	16.055	22.125
6	1.25	38.472	21.902	18.372	25.257
7	1.25	43.613	24.341	20.791	28.382
8	1.25	49.427	26.883	23.165	31.572
9	1.25	56.074	29.989	25.915	35.019
10	1.25	63.596	33.536	29.004	39.254
2	1	20.337	11.809	8.3075	13.751
3	1	25.823	14.965	11.188	17.928
4	1	30.929	18.378	13.98	21.187
5	1	36.24	21.963	16.735	24.712
6	1	43.089	25.778	19.739	29.258
7	1	51.051	30.135	23.169	34.187
8	1	60.801	35.349	26.987	39.962
9	1	72.875	42.239	31.853	46.777
10	1	87.927	51.049	37.906	55.871
2	0.75	20.354	12.425	8.1629	14.26
3	0.75	26.379	16.427	11.305	19.109
4	0.75	32.706	21.34	14.529	23.697
5	0.75	39.83	27.378	18.116	29.236
6	0.75	50.019	35.298	22.73	37.049
7	0.75	63.144	46.085	28.723	46.944
8	0.75	81.142	61.013	36.548	60.802
9	0.75	105.95	83.406	48.016	80.807
10	0.75	140.94	117.15	64.668	112.58
2	0.5	20.324	14.128	8.3063	15.371
3	0.5	26.836	20.176	11.817	22.289
4	0.5	34.756	28.623	15.726	30.462
5	0.5	44.566	41.845	21.031	42.603
6	0.5	60.052	64.459	29	63.123
7	0.5	82.596	104.93	42.236	96.359
8	0.5	117.92	178.45	63.079	162.51
9	0.5	173.5	322.42	100.4	302.3
10	0.5	264.17	602.91	166.56	596.41

Note: *m* denotes embedding dimension, *r* is defined in terms of standard deviation of data. BDS statistics asymptotically have the standard normal distribution. All estimates are significant at 1% significance level, implying that null hypothesis of iid is rejected in raw return data.

Table 2(b). BDS test: ER returns, AR-GARCH filtered standardized residuals

m	*r*	CAD/USD	JPY/USD	CHF/USD	USD/BP
2	1.5	0.6419	0.87727	-0.92708	0.43119
3	1.5	0.48505	1.7565	-1.0344	1.0354
4	1.5	0.16398	2.78**	-0.74492	1.3761
5	1.5	0.13854	3.4908**	-0.48979	1.5118
6	1.5	0.34081	3.8319**	-0.15693	2.0064*
7	1.5	0.65996	4.1633**	0.23771	2.4265*
8	1.5	0.9761	4.297**	0.43288	2.7032**
9	1.5	1.2832	4.369**	0.76466	3.0028**
10	1.5	1.4349	4.4058**	0.98506	3.4542**
2	1.25	0.56931	1.0756	-1.0497	1.0245
3	1.25	0.44311	1.9367	-1.0483	1.8317
4	1.25	0.22446	3.0278**	-0.68367	2.3022*
5	1.25	0.27725	3.899**	-0.34035	2.6417**
6	1.25	0.60211	4.483**	0.11795	3.4333**
7	1.25	0.9778	5.0101**	0.65682	4.1369**
8	1.25	1.3533	5.344**	0.96243	4.7316**
9	1.25	1.7087	5.693**	1.3941	5.367**
10	1.25	1.8908	6.0283**	1.7143	6.2911**
2	1	0.55582	1.4251	-1.0983	1.8706
3	1	0.46478	2.501*	-0.99348	2.9624**
4	1	0.32575	3.7505**	-0.53886	3.6883**
5	1	0.37075	4.9376**	-0.12642	4.4258**
6	1	0.7899	5.9887**	0.4577	5.8202**
7	1	1.2025	6.9732**	1.1141	7.2224**
8	1	1.6494	7.7562**	1.5202	8.7132**
9	1	2.0899*	8.7841**	2.0975*	10.51**
10	1	2.3608*	9.9794**	2.552*	12.999**
2	0.75	0.51676	2.0241*	-1.0197	2.8989**
3	0.75	0.40516	3.4567**	-0.82193	4.4374**
4	0.75	0.31871	5.0132**	-0.2934	5.5733**
5	0.75	0.36807	6.7554**	0.28228	7.1955**
6	0.75	0.93333	8.7419**	0.9585	10.128**
7	0.75	1.5535	10.895**	1.8342	13.681**
8	0.75	2.1067*	13.063**	2.5027*	18.891**
9	0.75	2.6548**	16.152**	3.3321**	27.099**
10	0.75	3.1692**	20.194**	4.061**	39.848**
2	0.5	0.57964	2.9242**	-0.83043	4.1239**
3	0.5	0.51216	4.8708**	-0.56699	6.7124**
4	0.5	0.46025	7.0359**	-0.07167	9.2376**
5	0.5	0.44559	9.8028**	0.49499	13.652**
6	0.5	1.209	13.887**	1.2106	22.506**
7	0.5	2.1979*	19.781**	2.5425*	38.577**
8	0.5	2.6535**	28.011**	3.4042**	73.037**
9	0.5	2.9484**	41.988**	4.3819**	149.14**
10	0.5	4.0107**	65.505**	5.6417**	321.82**

Note: *m* denotes embedding dimension, *r* is defined in terms of standard deviation of data. Asymptotic standard normal distribution was used for the hypothesis testing.
* Significant at 5% level. ** Significant at 1% level.

Table 3. Estimates of the correlation dimension for Henon map

Embedding dimension	Correlation dimension
2	1.1926
3	1.2872
4	1.2256
5	1.2018
6	1.2125
7	1.2157
8	1.2147
9	1.2174
10	1.2332
11	1.2254
12	1.2224

Table 4. Estimates of correlation dimension for some exchange rate returns

Embedding dimension	JPY/USD Raw returns	USD/BP AR-GARCH residuals
2	1.6310	1.8434
3	2.3548	2.7425
4	2.9782	3.6040
5	3.4965	4.4105
6	3.9298	5.0320
7	4.2813	5.3985
8	4.5744	5.7281
9	4.8087	5.9318
10	4.9867	5.9536
11	5.1608	5.8975
12	5.2958	5.8640

Table 5. Estimates of maximum Lyapunov exponent for Henon map

Embedding dimension	Maximum Lyapunov exponent
2	0.4195
3	0.4192
4	0.4189
5	0.4185
6	0.4181
7	0.4170
8	0.4141
9	0.4089
10	0.4020
11	0.3907
12	0.3755

Figure 1. Daily exchange rates

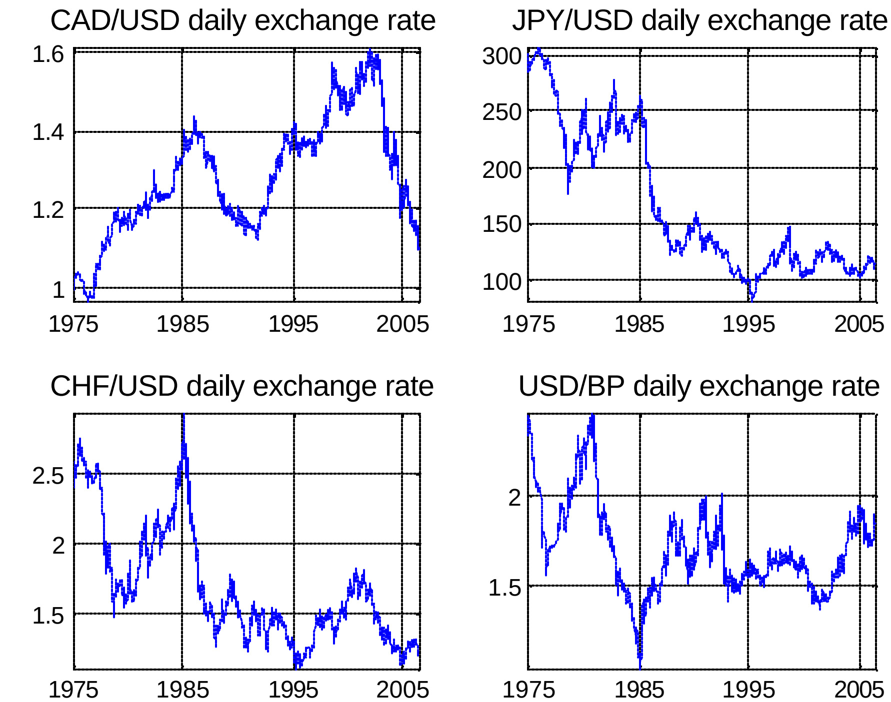


Figure 2. Exchange rate returns

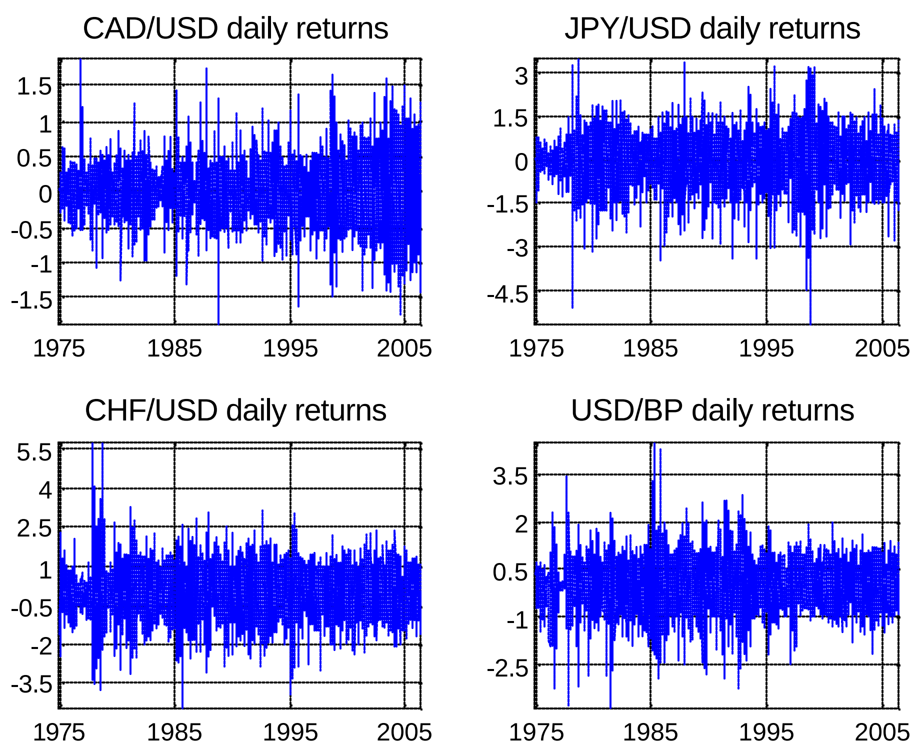


Figure 3. Normalized exchange rates

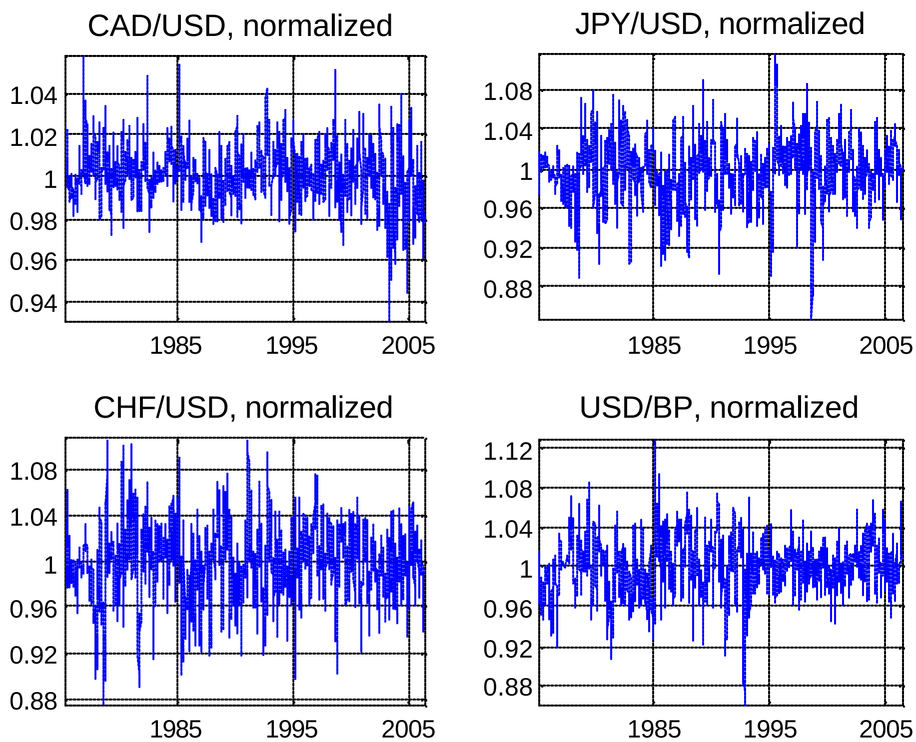


Figure 4. Exchange rate volatility

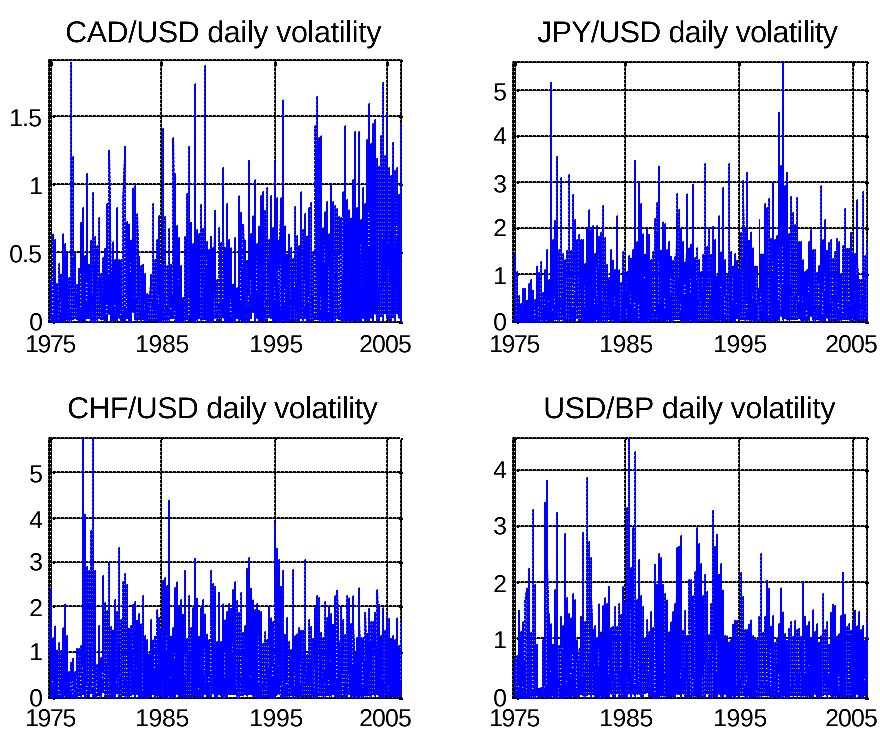


Figure 5 (a). Autocorrelation function, CAD/USD variables

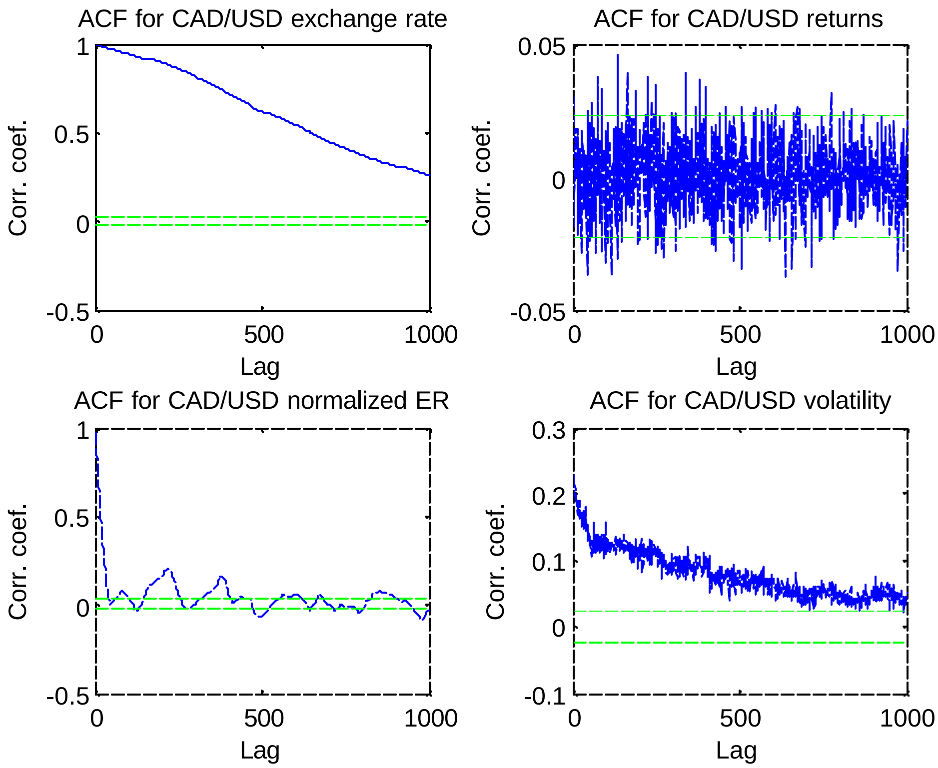


Figure 5 (b). Autocorrelation function, JPY//USD variables

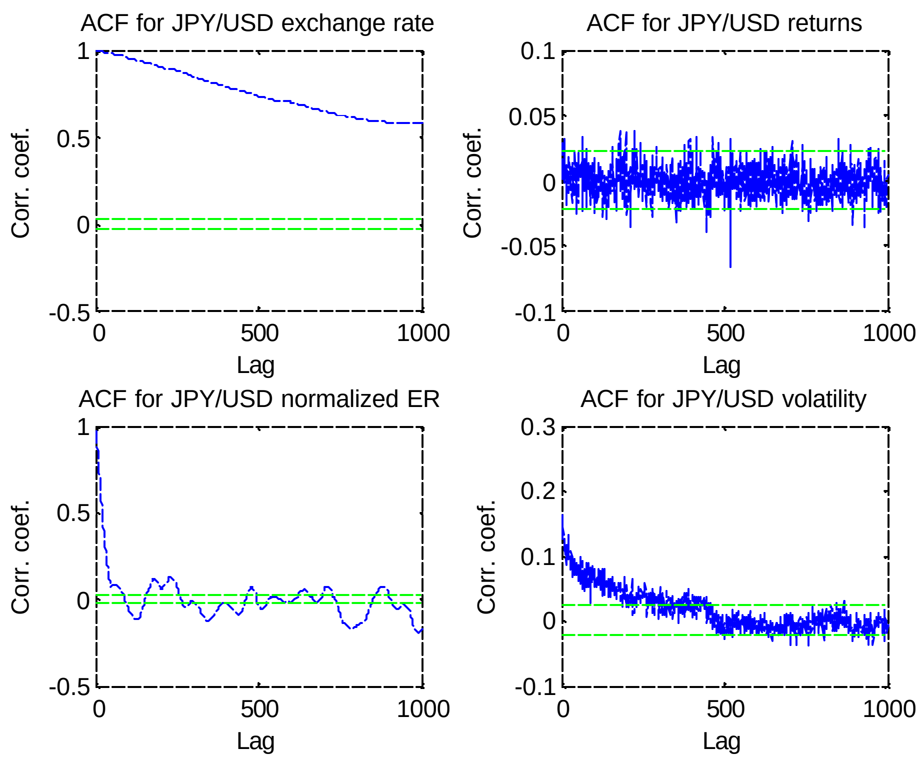


Figure 5 (c). Autocorrelation function, CHF//USD variables

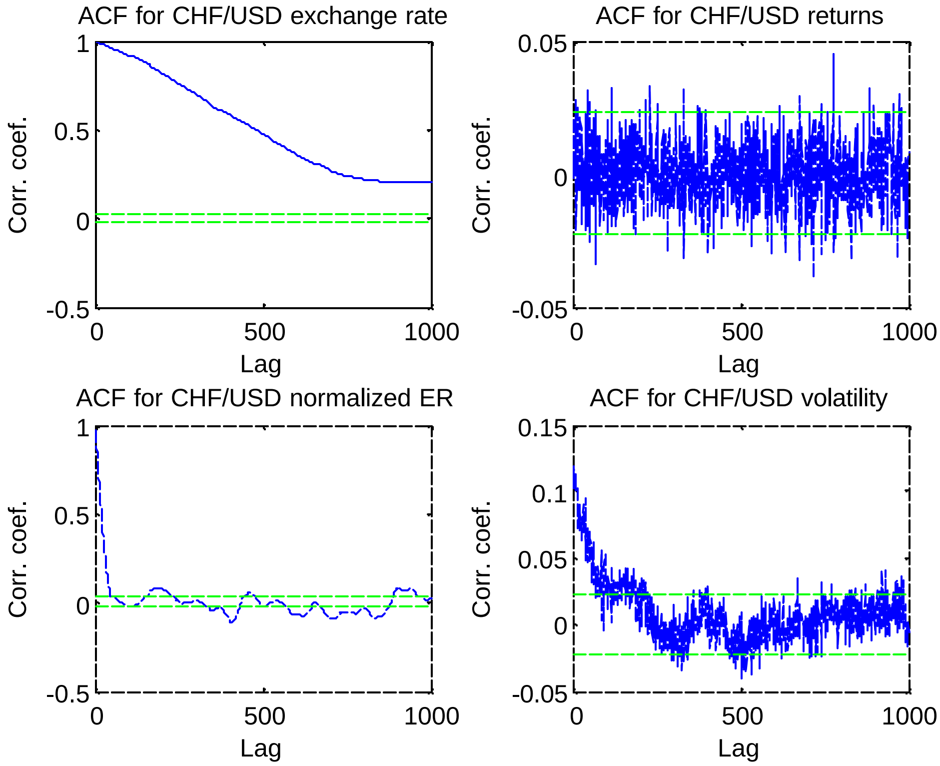


Figure 5 (d). Autocorrelation function, USD/BP variables

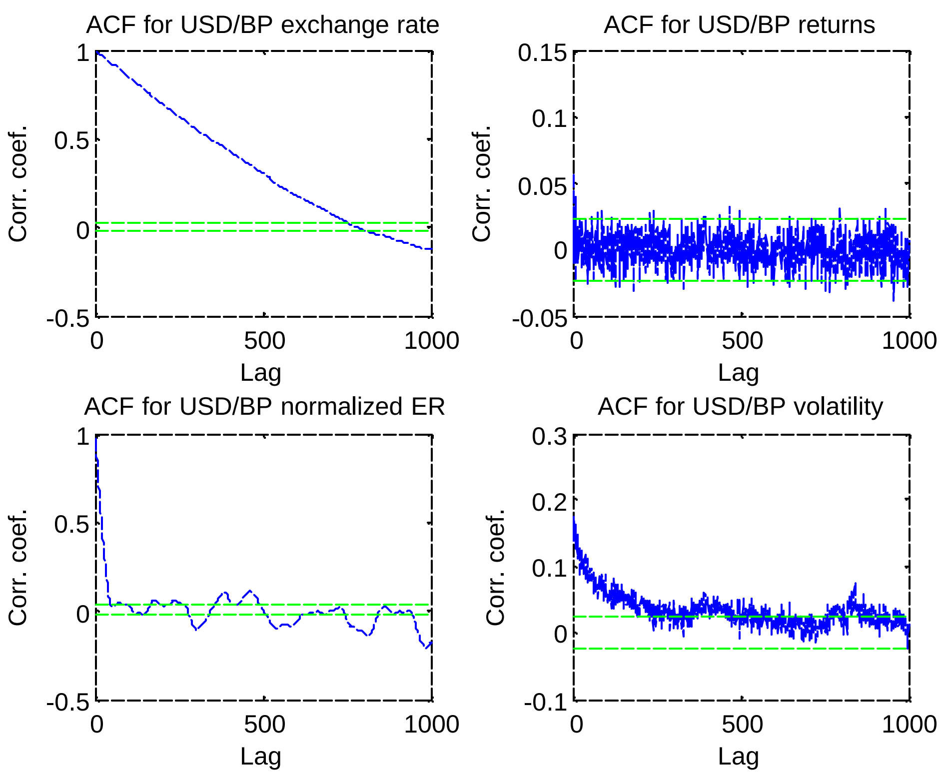


Figure 6 (a). Phase portraits for exchange rate levels

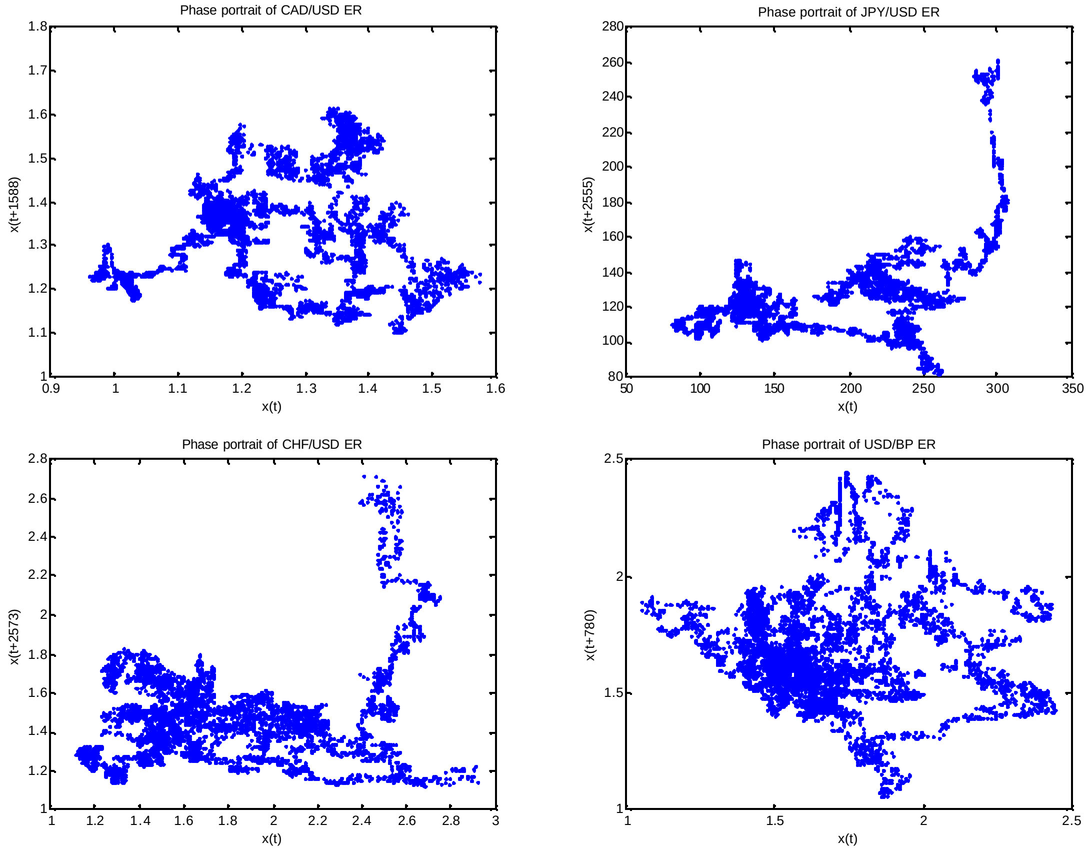


Figure 6 (b). Phase portraits for exchange rate returns

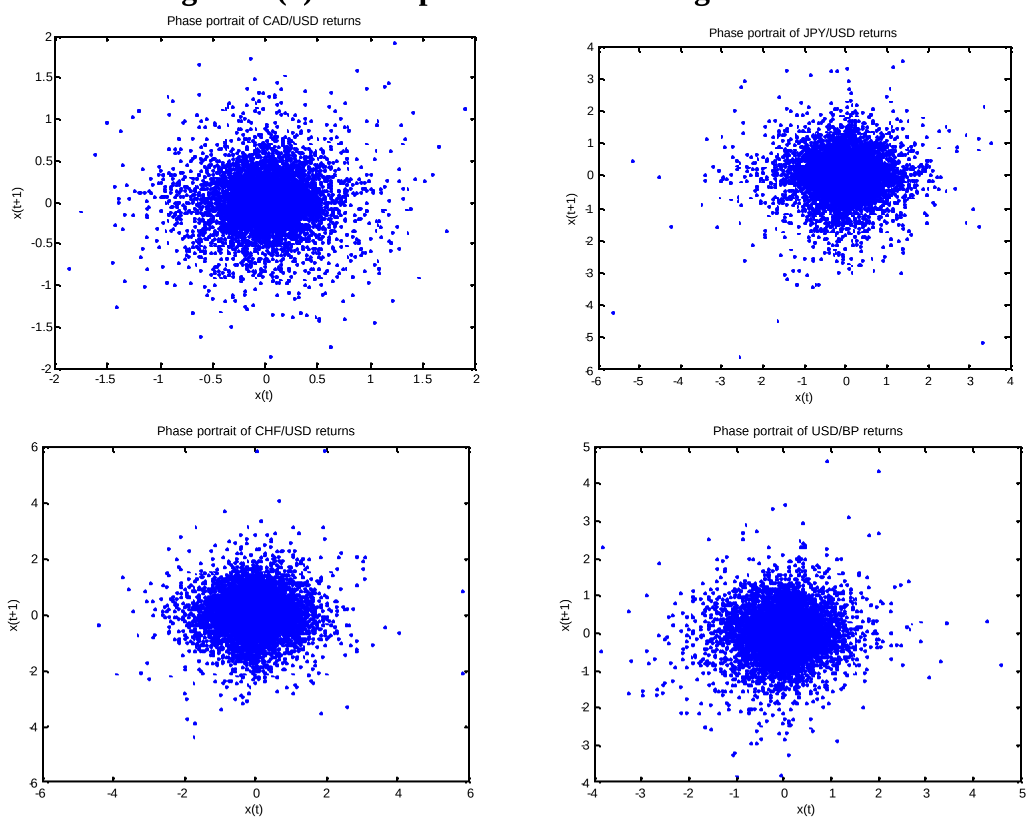


Figure 6 (c). Phase portraits for normalized exchange rates

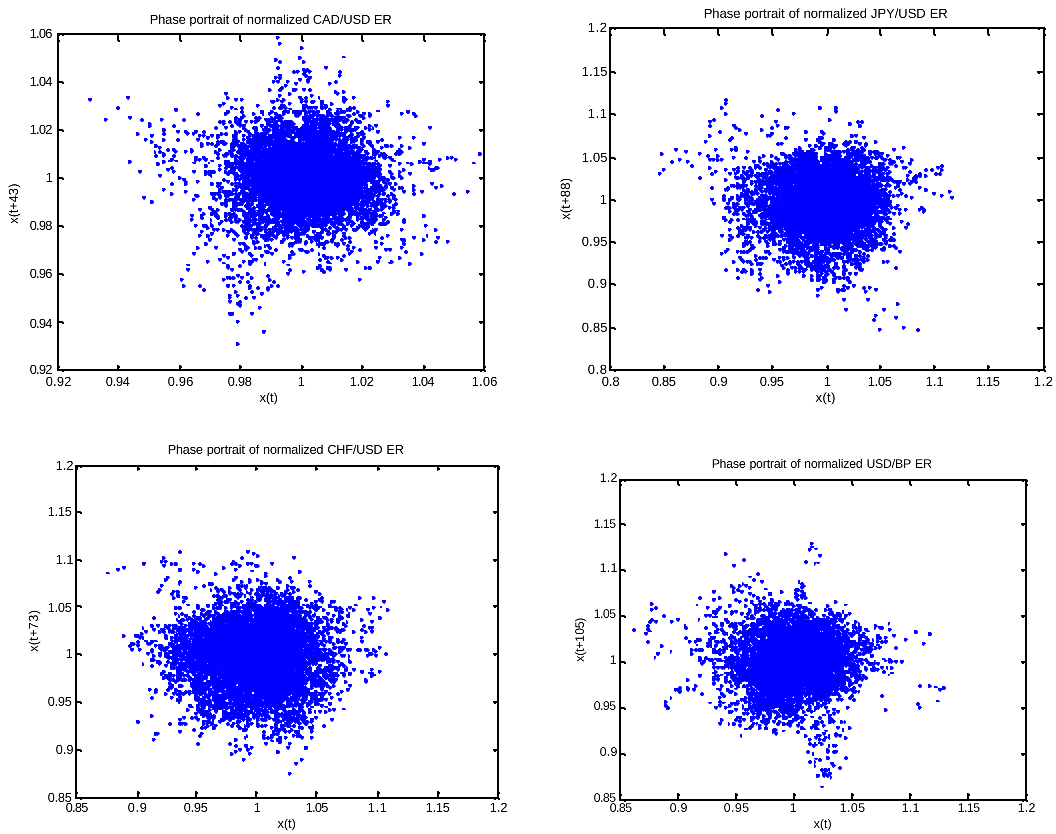


Figure 6 (d). Phase portraits for exchange rate volatilities

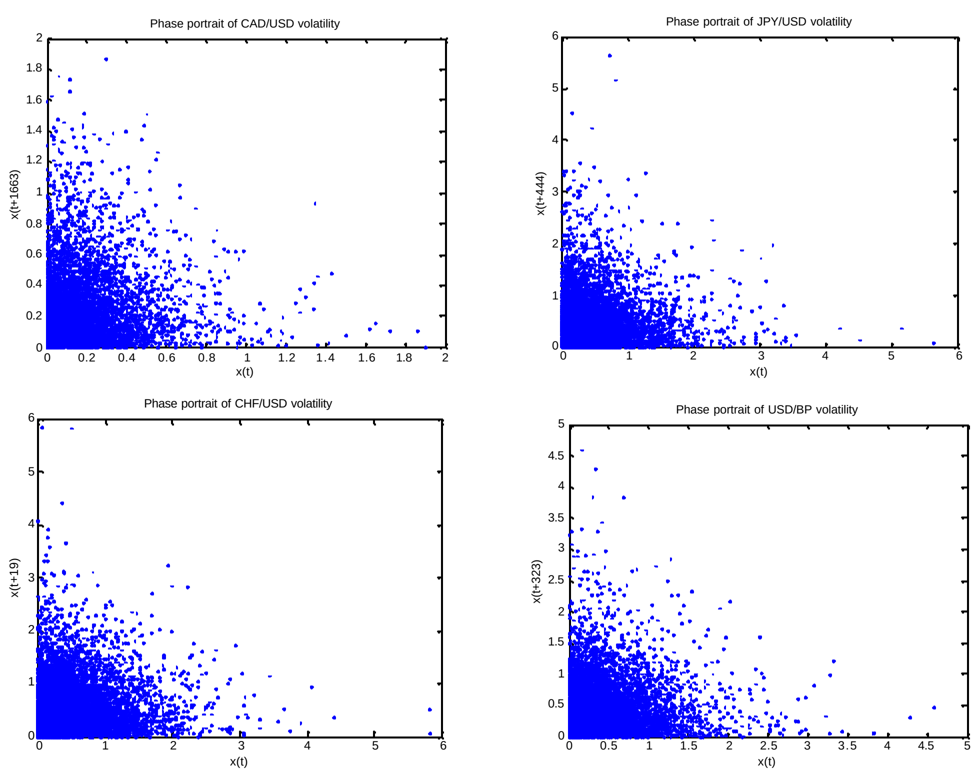


Figure 7. Henon Map

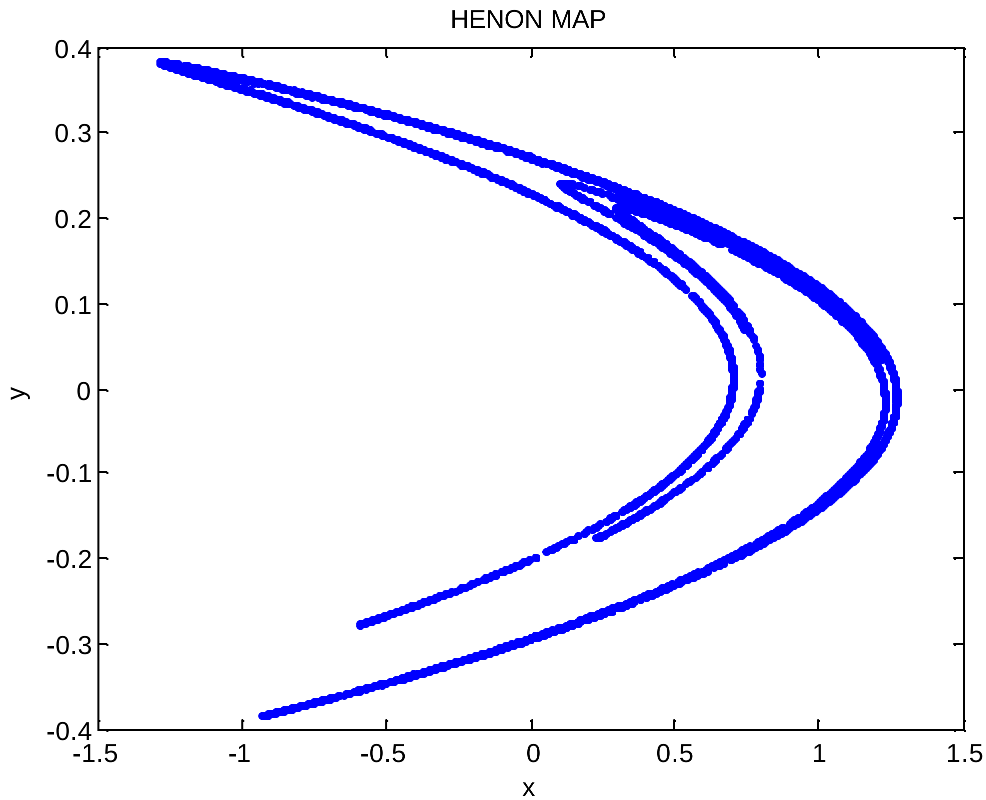
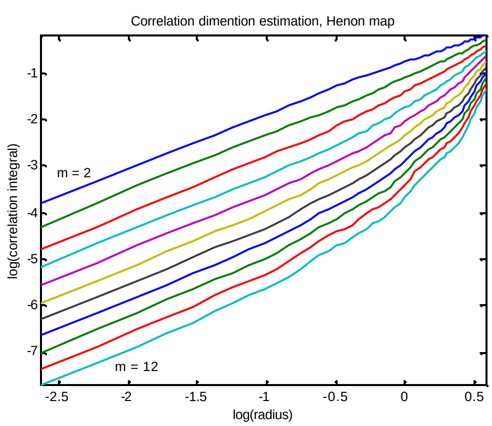


Figure 8. Estimation of correlation dimension for Henon map



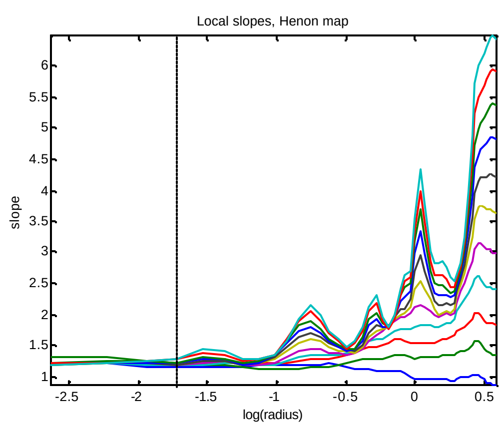


Figure 9 (a). Estimation of correlation dimension CAD/USD returns

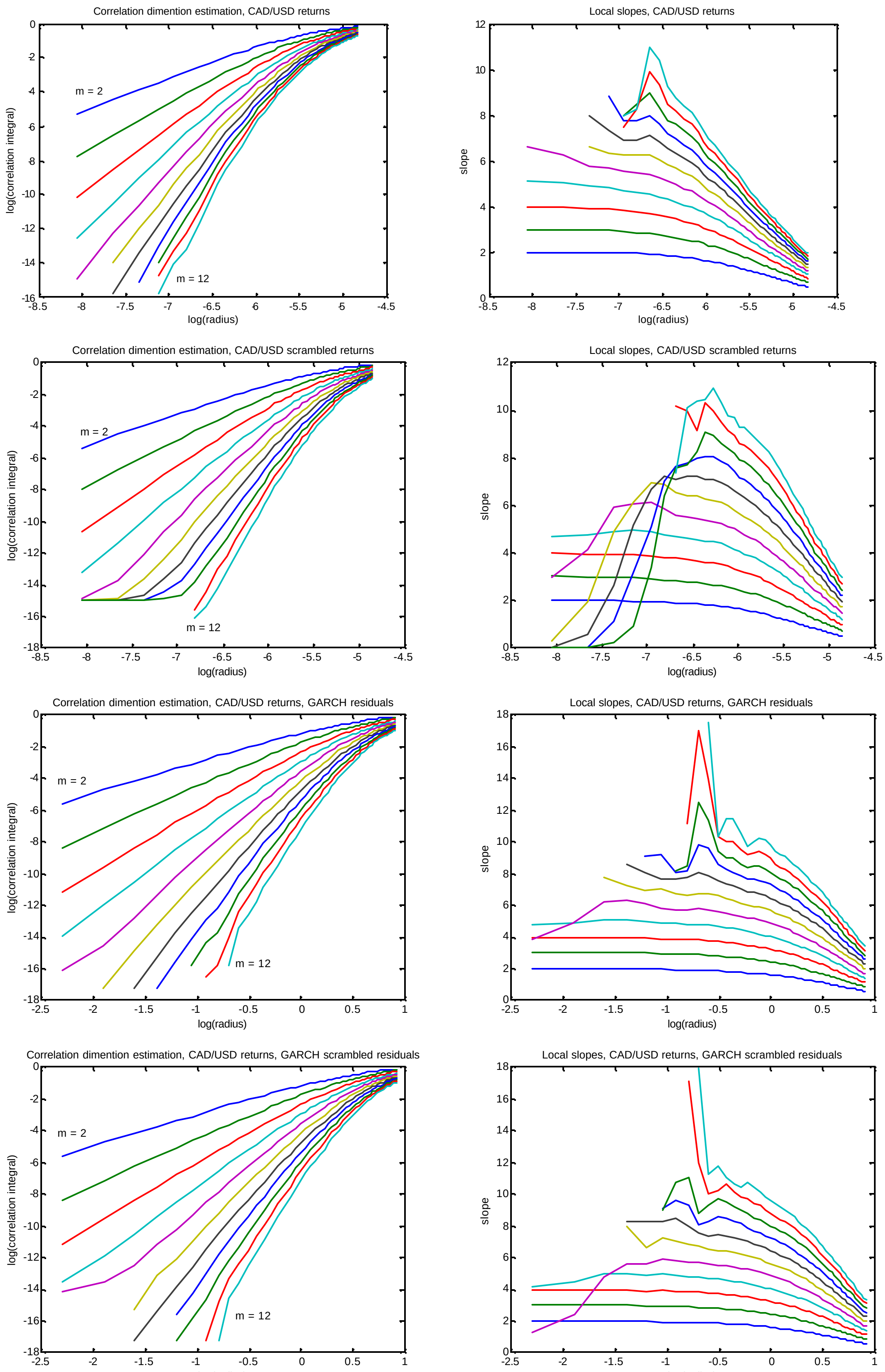


Figure 9 (b). Estimation of correlation dimension JPY/USD returns

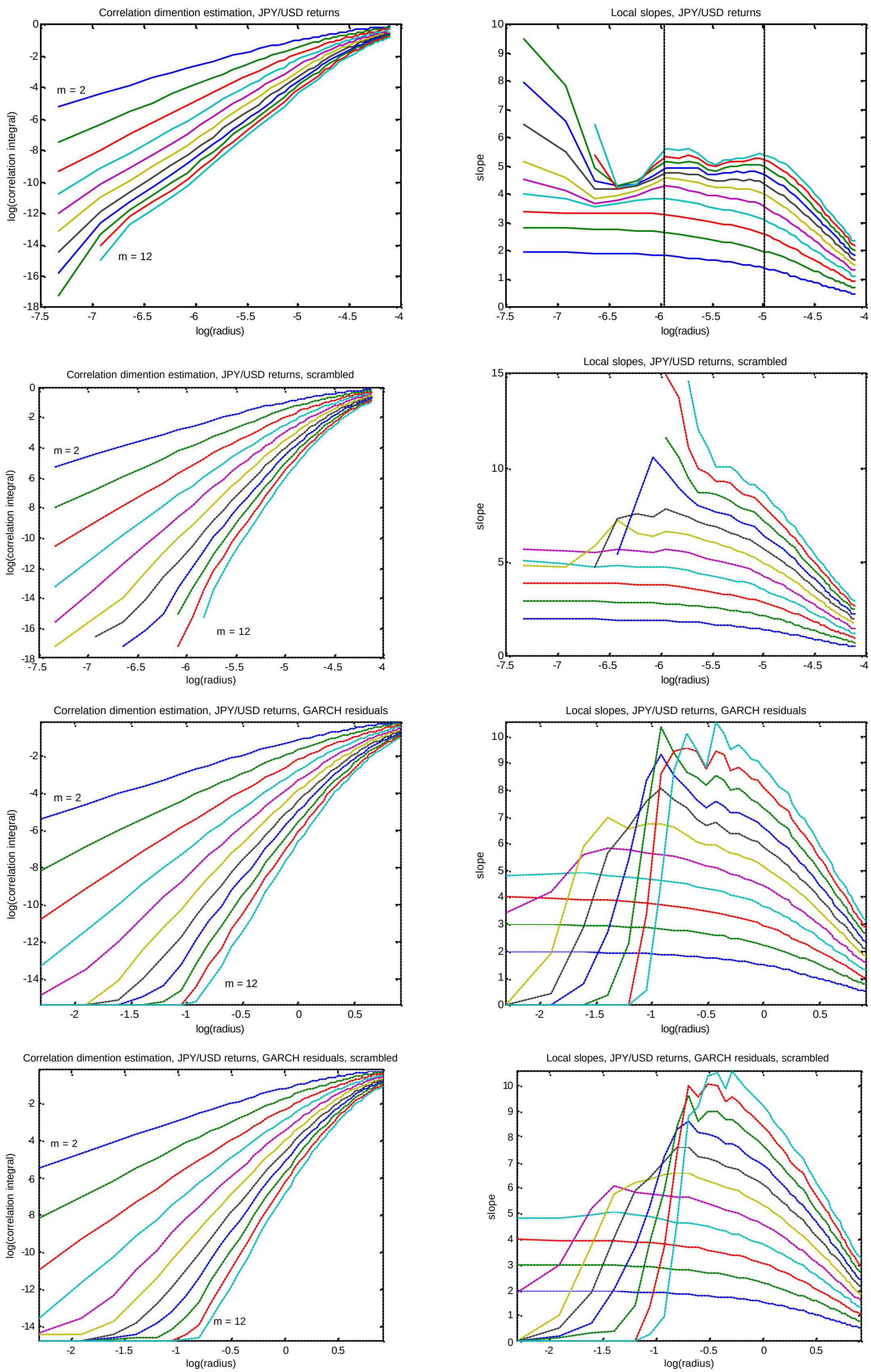


Figure 9 (c). Estimation of correlation dimension CHF/USD returns

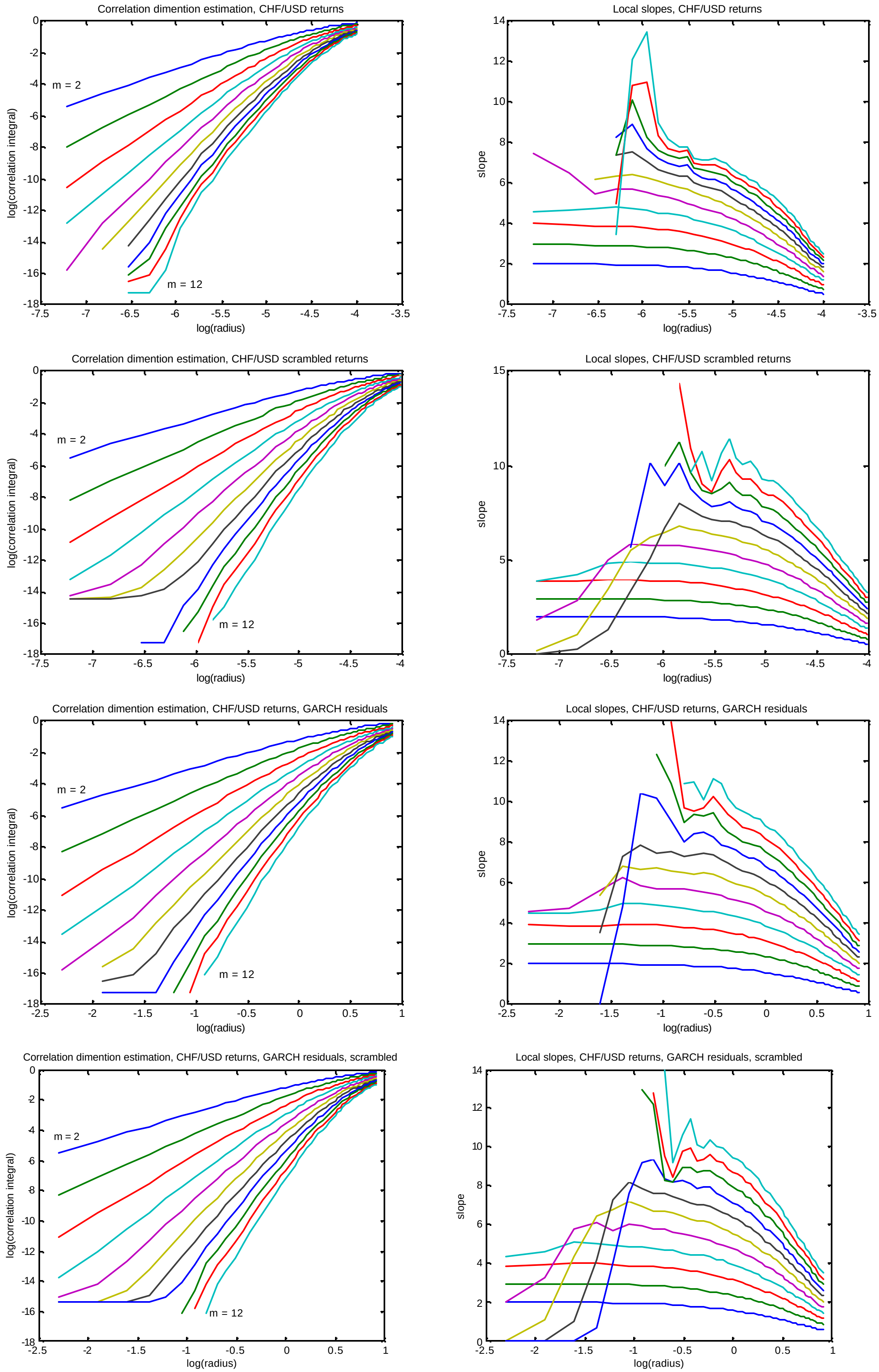


Figure 9 (d). Estimation of correlation dimension USD/BP returns

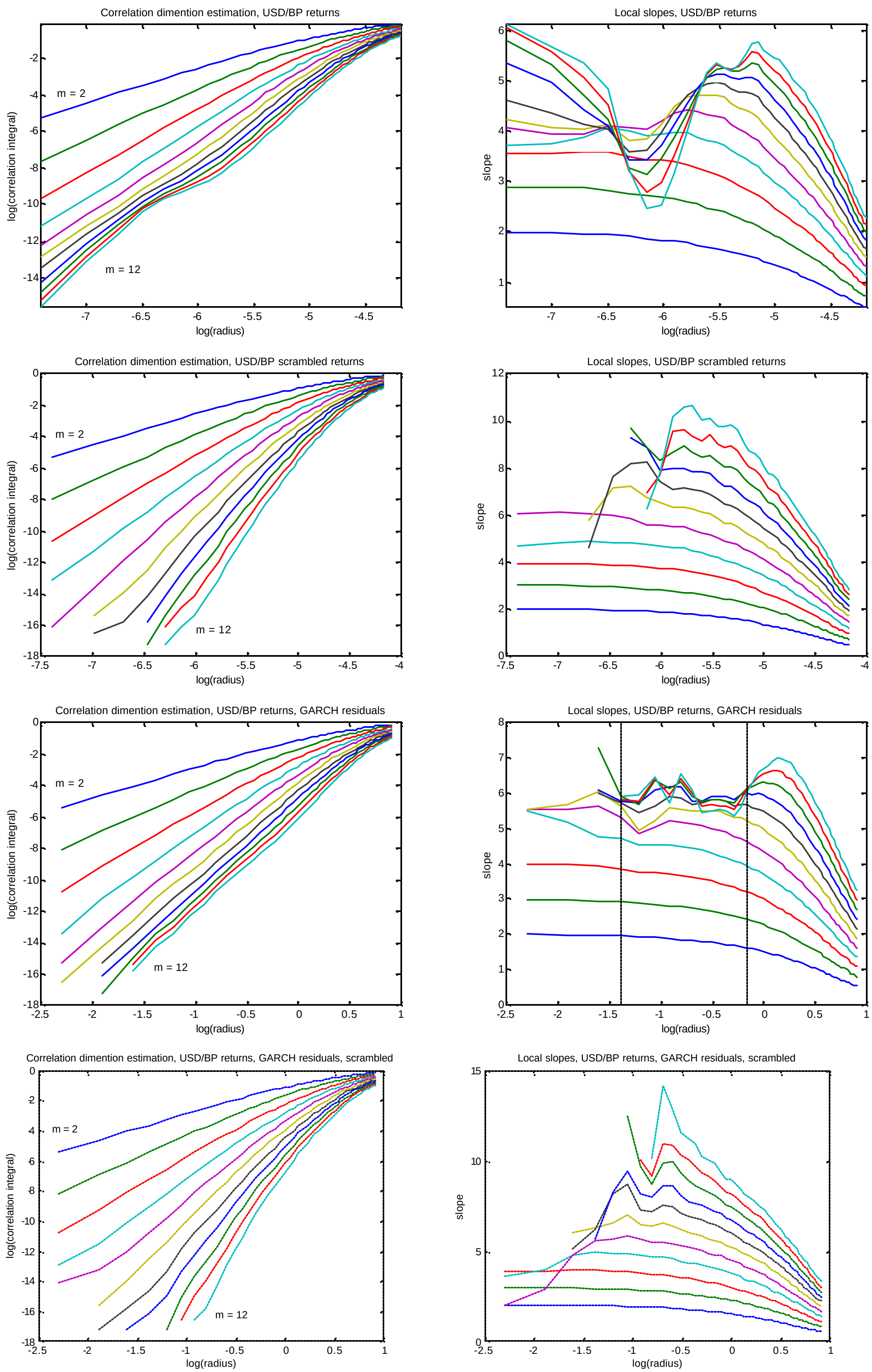


Figure 10 (a). Estimation of correlation dimension for CAD/USD normalized ER

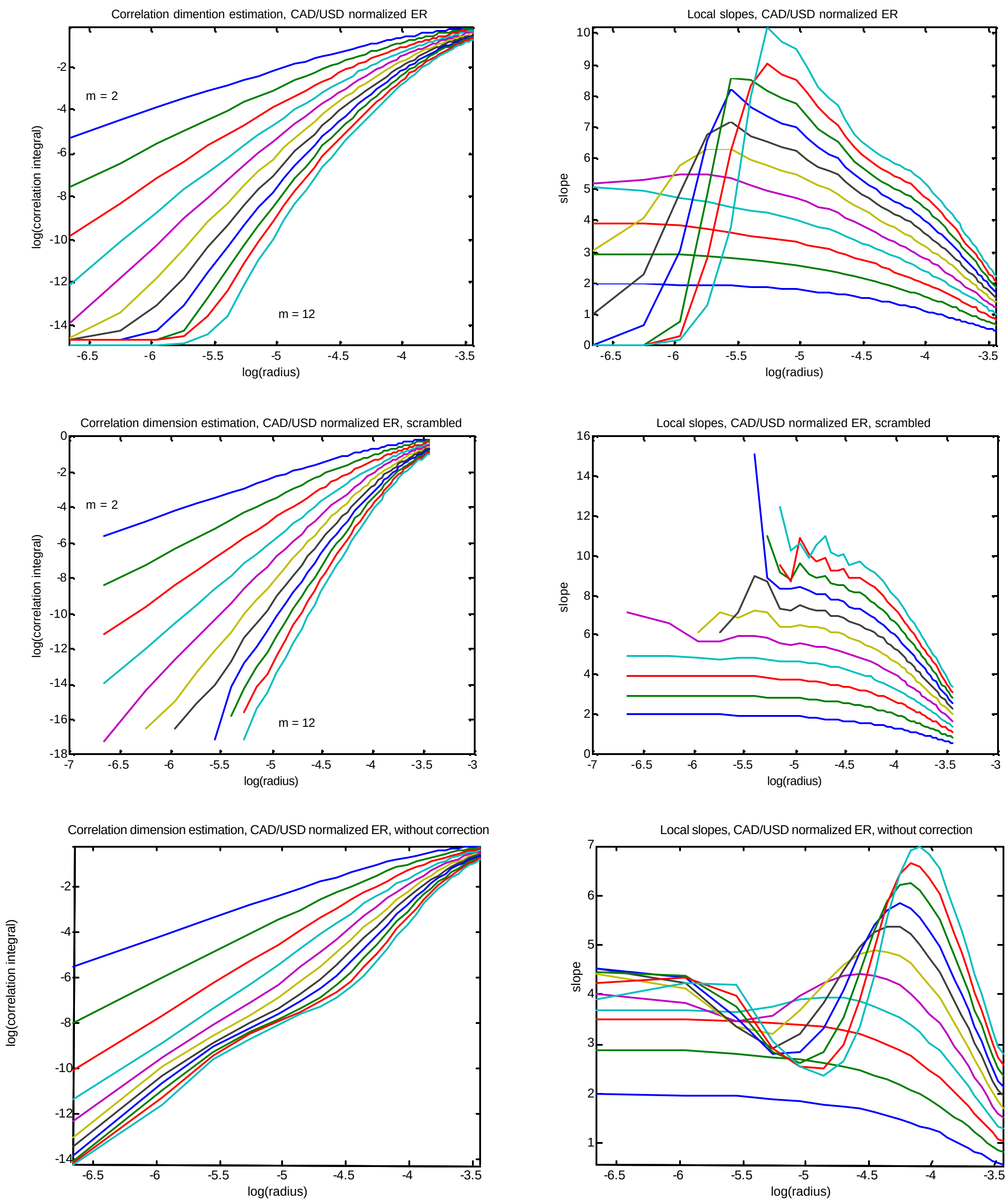


Figure 10 (b). Estimation of correlation dimension for JPY/USD normalized ER

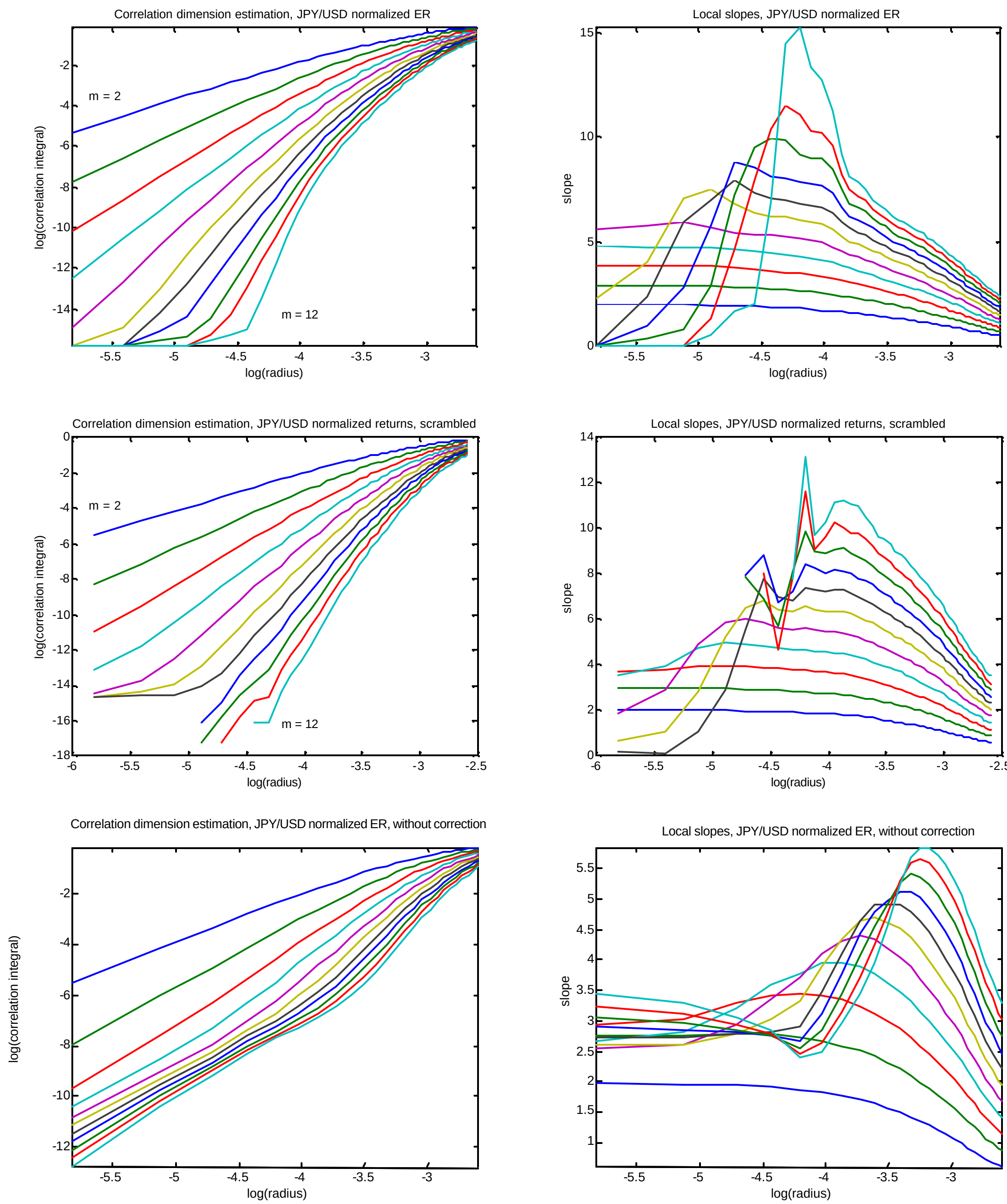


Figure 10 (c). Estimation of correlation dimension for CHF/USD normalized ER

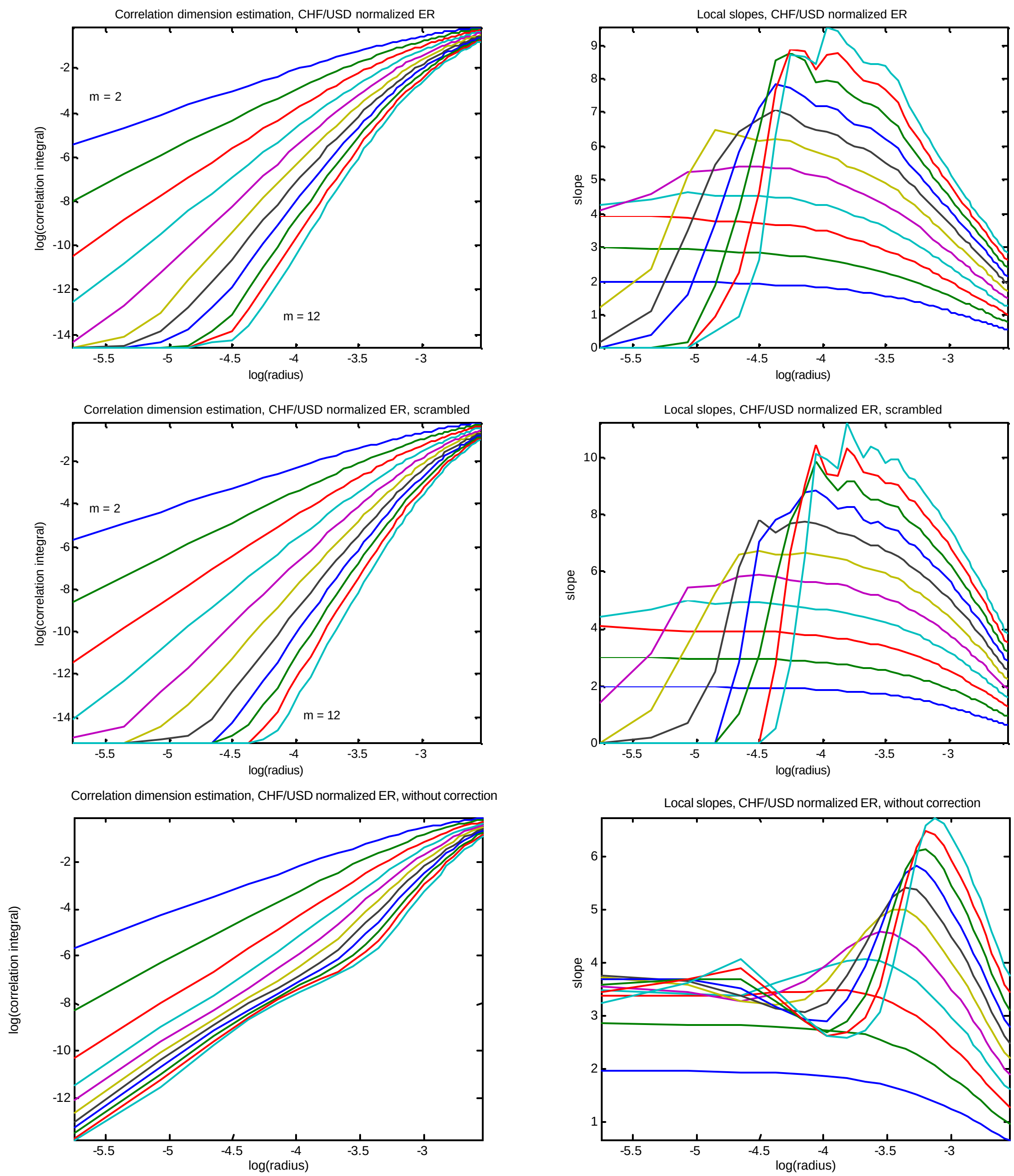


Figure 10 (d). Estimation of correlation dimension for USD/BP normalized ER

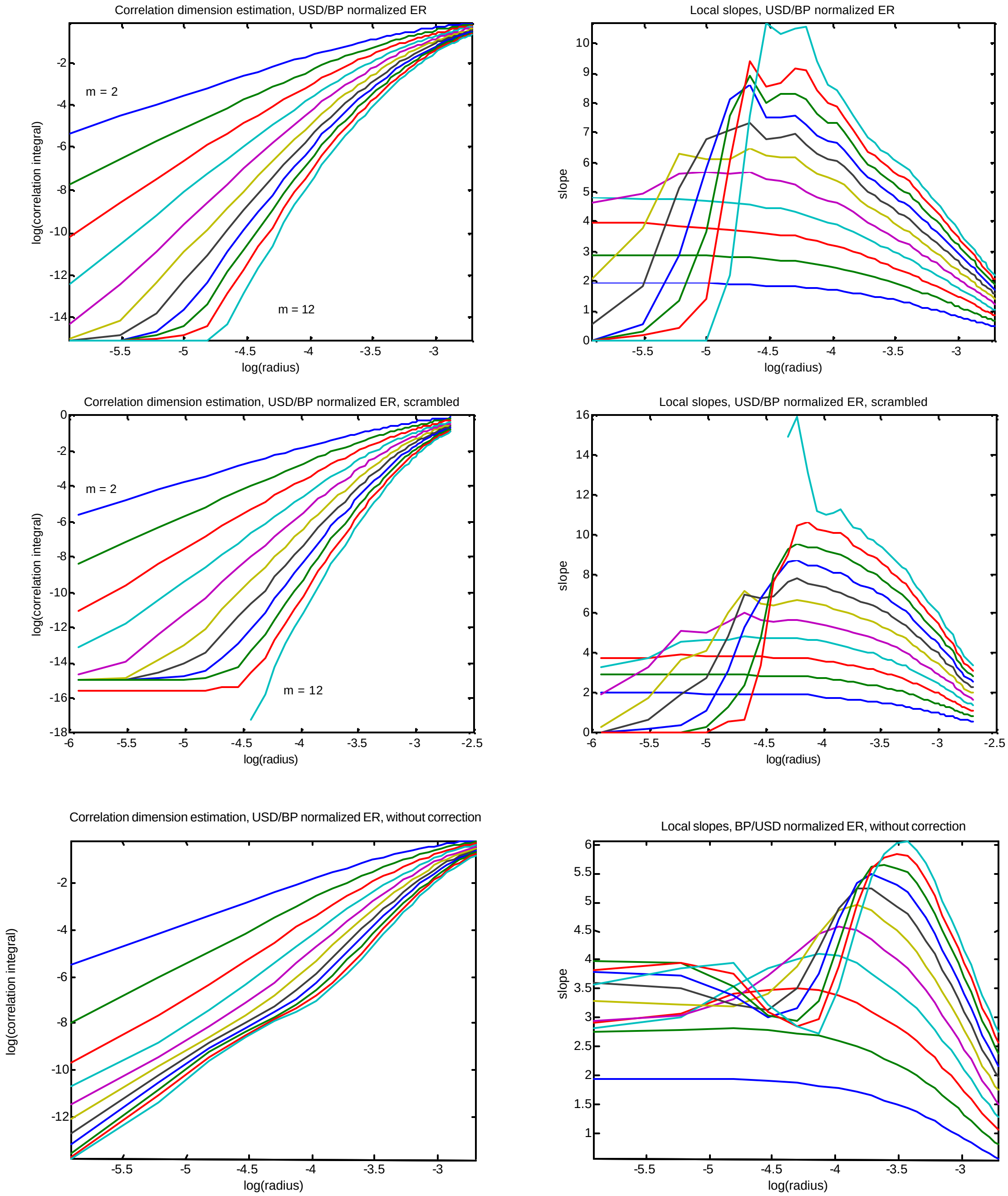


Figure 11 (a). Estimation of correlation dimension for CAD/USD volatility

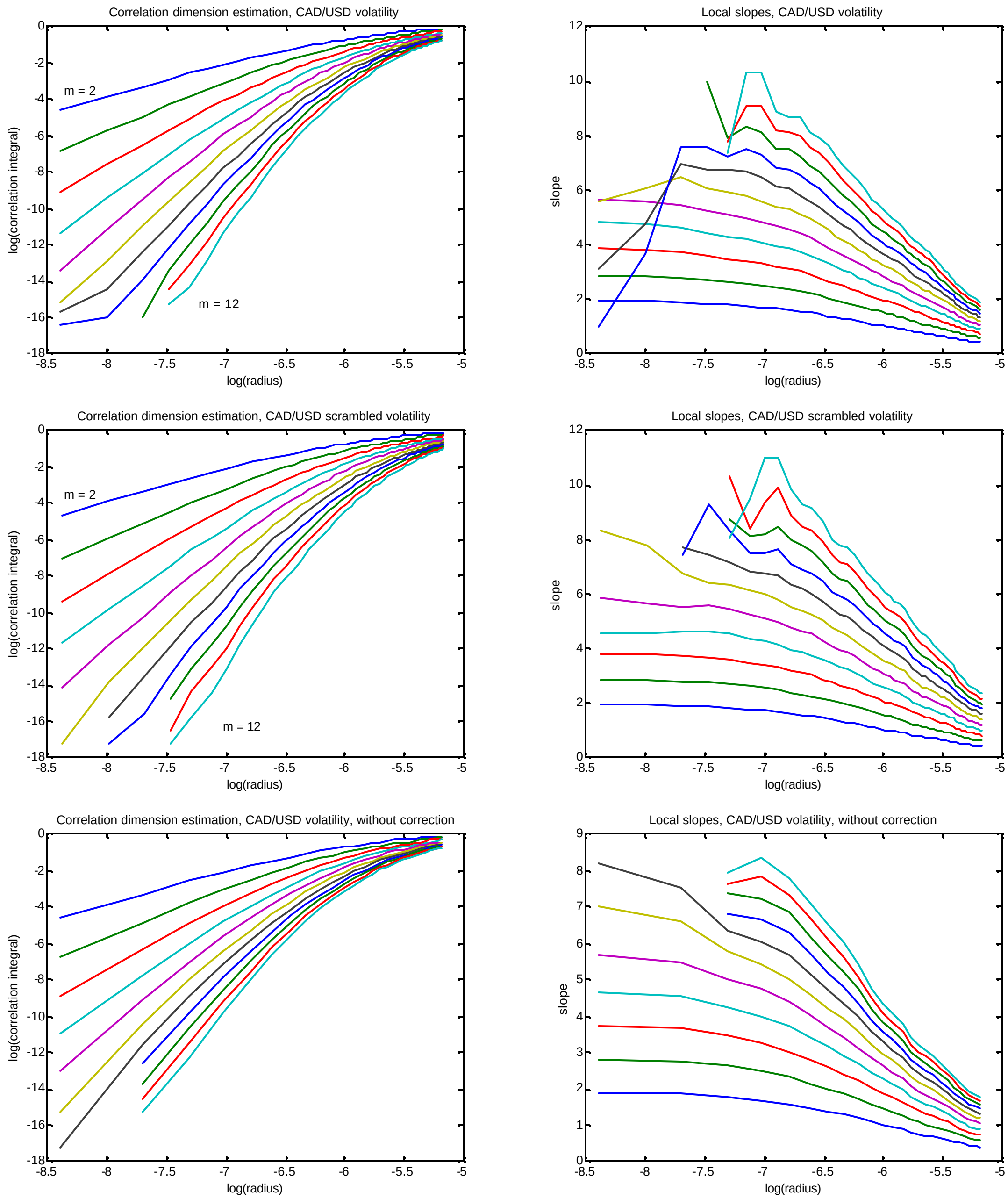


Figure 11 (b). Estimation of correlation dimension for JPY/USD volatility

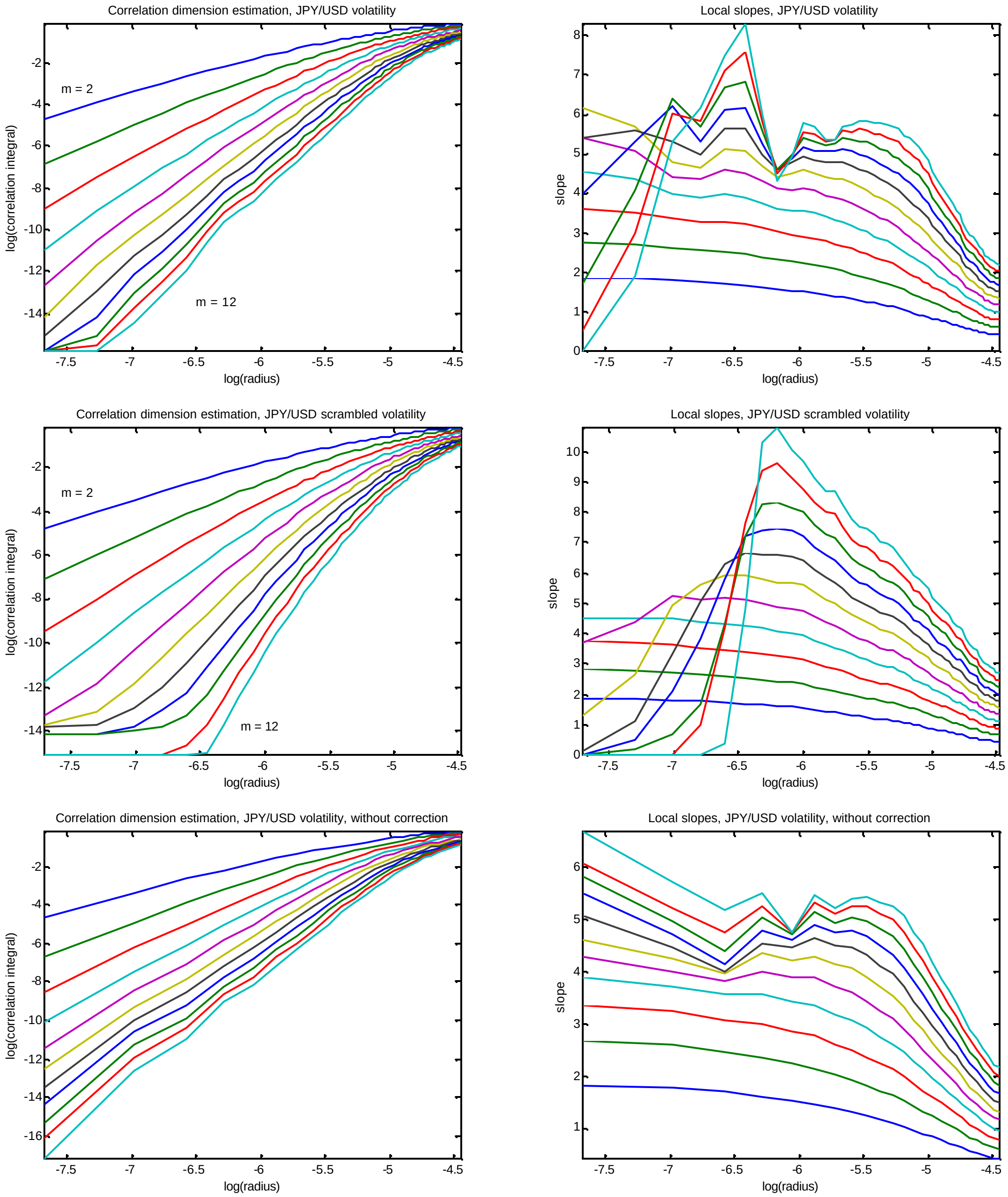


Figure 11 (c). Estimation of correlation dimension for CHF/USD volatility

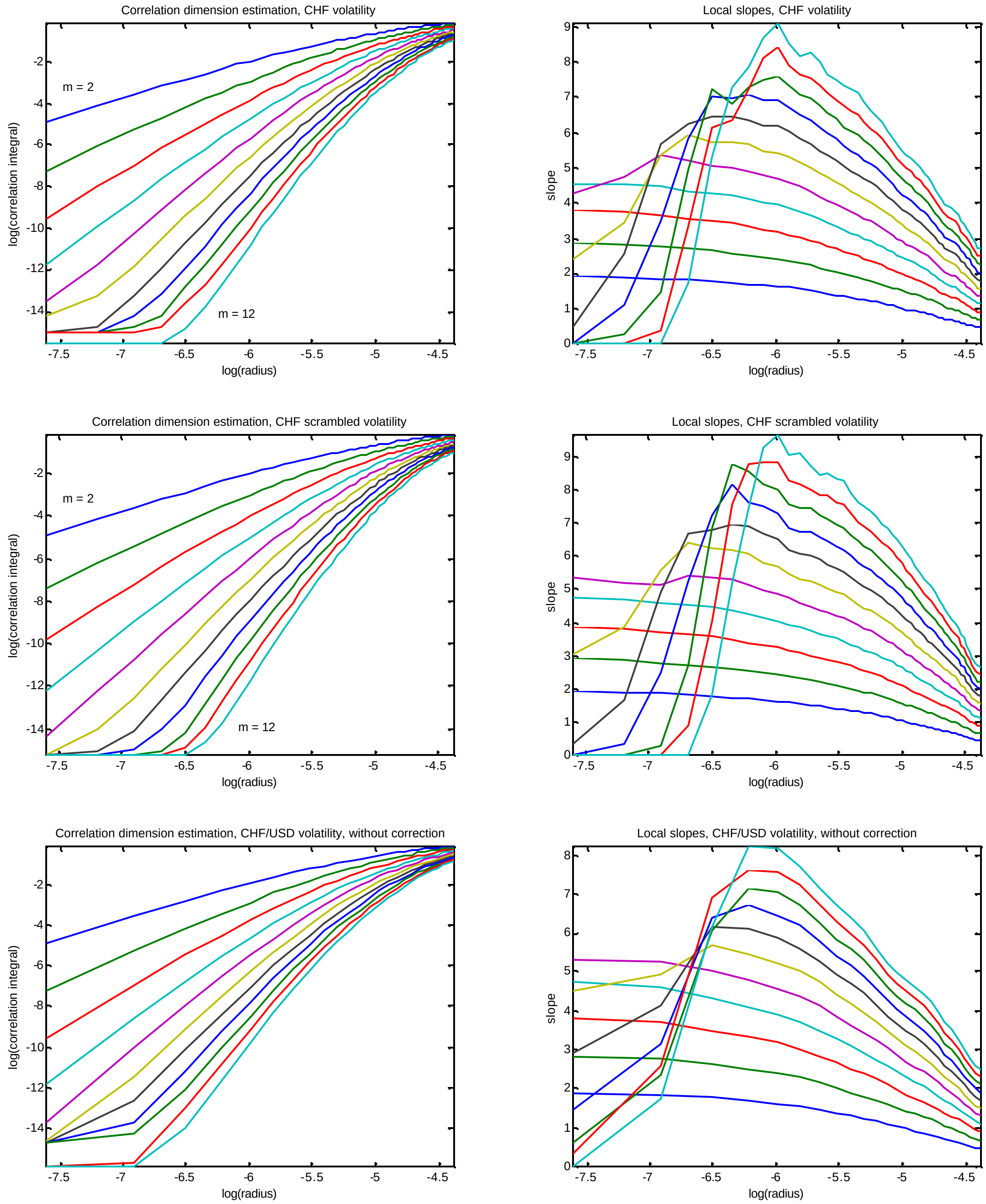


Figure 11 (d). Estimation of correlation dimension for USD/BP volatility

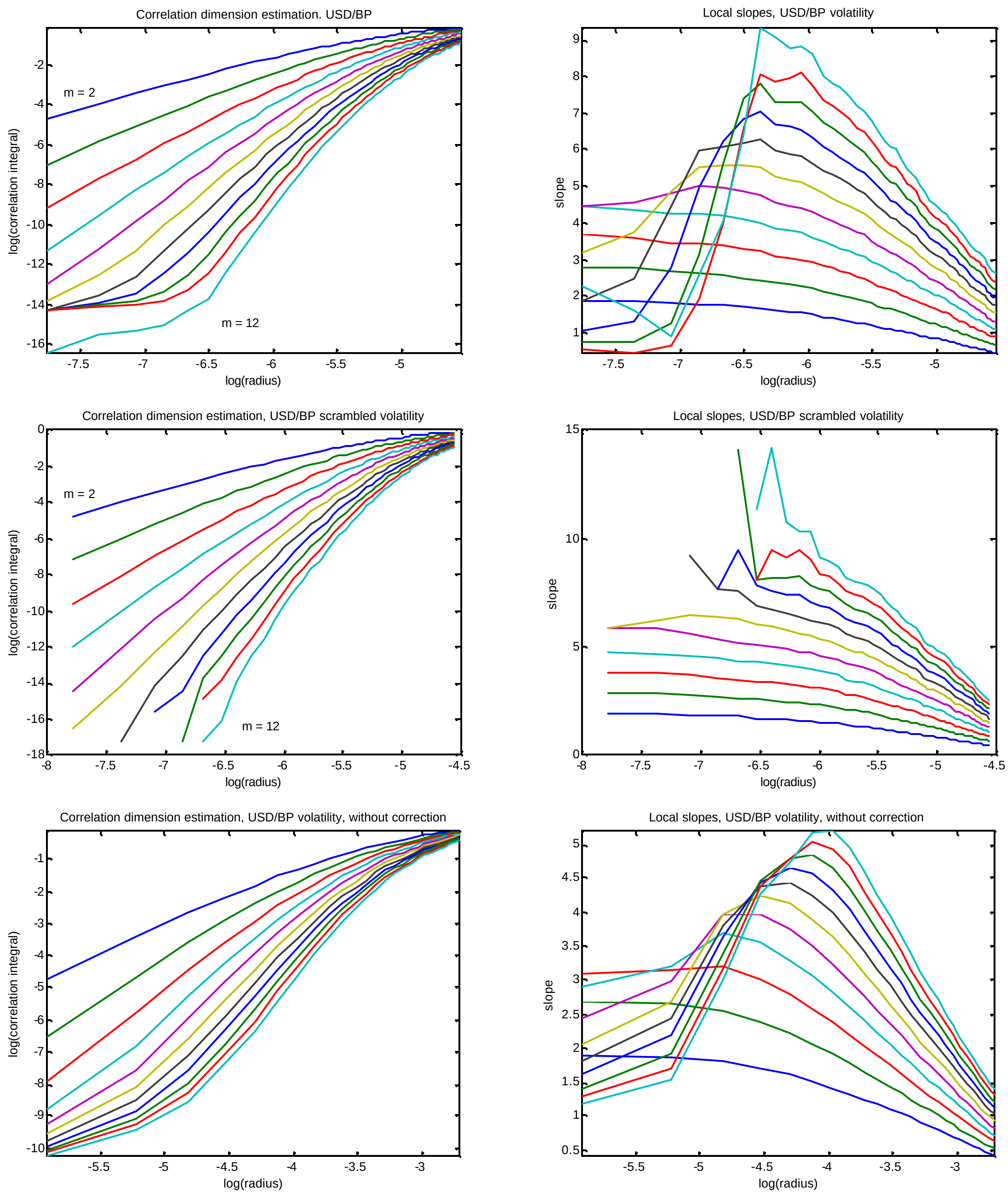


Figure 12. Maximum Lyapunov exponent estimation, Henon map

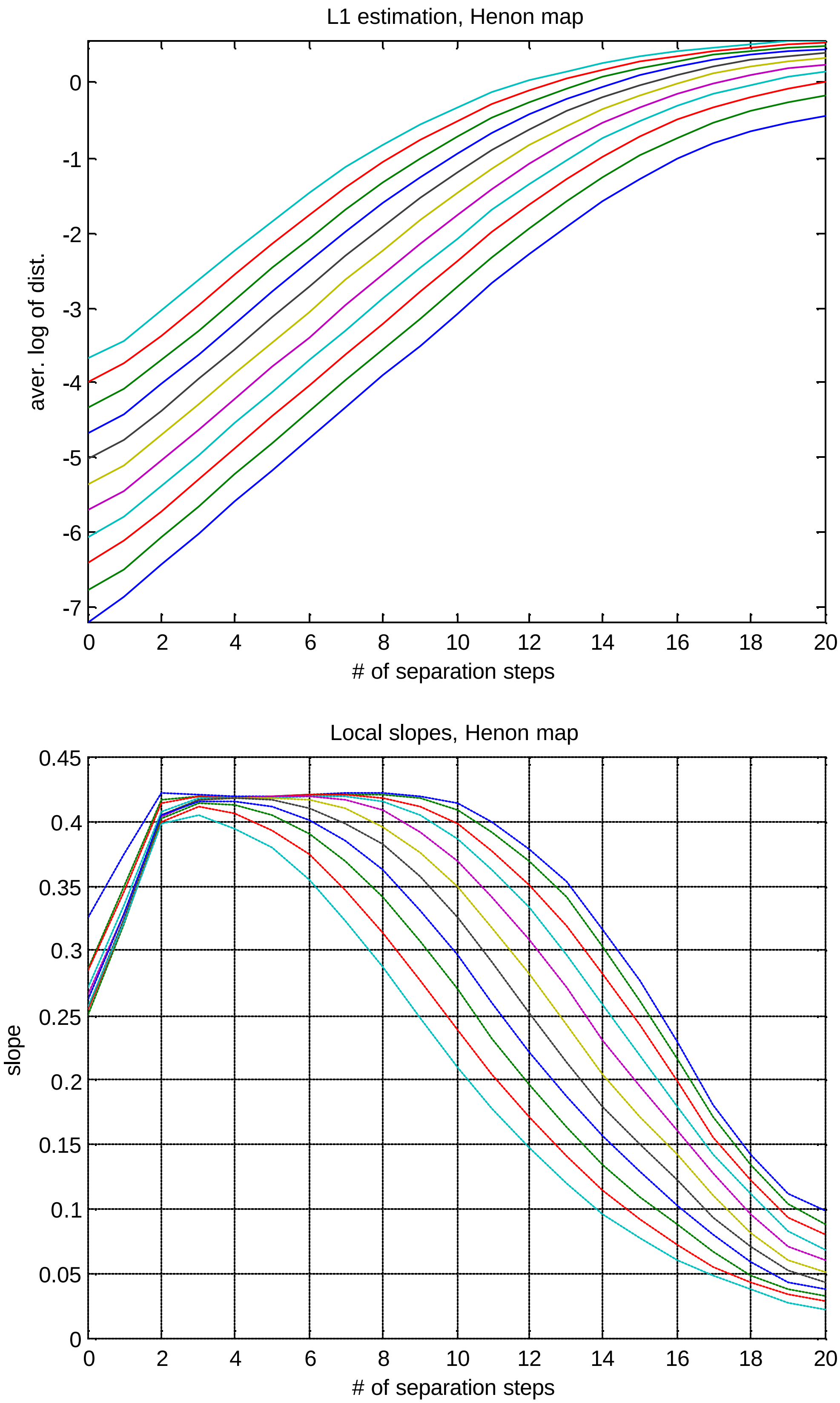


Figure 13. Maximum Lyapunov exponent estimation, JPY/USD returns

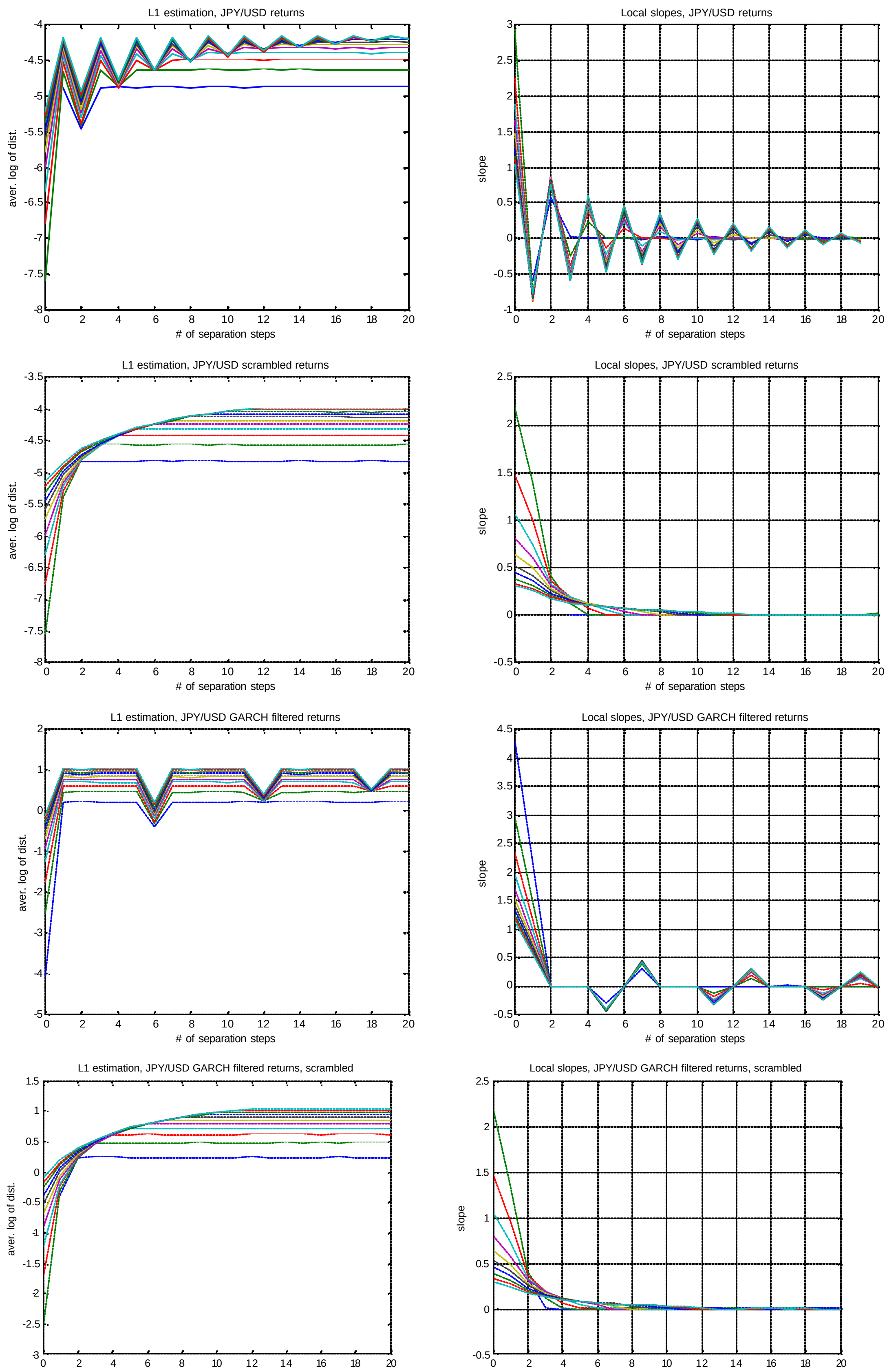


Figure 14. MLE estimation, JPY/USD normalized ER and volatility